Abstract. Let $H_0$ denote the kernel of the endomorphism, defined by $z \mapsto (z/\pi)^2$, of the real algebraic group given by the Weil restriction of $\mathbb{C}^*$. Let $X$ be a nondegenerate anisotropic conic in $\mathbb{P}_\mathbb{R}^2$. The principal $\mathbb{C}^*$–bundle over the complexification $X_\mathbb{C}$, defined by the ample generator of $\text{Pic}(X_\mathbb{C})$, gives a principal $H_0$–bundle $F_{H_0}$ over $X$ through a reduction of structure group. Given any principal $G$–bundle $E_G$ over $X$, where $G$ is any connected reductive linear algebraic group defined over $\mathbb{R}$, we prove that there is a homomorphism $\rho : H_0 \rightarrow G$ such that $E_G$ is isomorphic to the principal $G$–bundle obtained by extending the structure group of $F_{H_0}$ using $\rho$.

1. Introduction

Let $F_{\mathbb{C}^*}$ denote the principal $\mathbb{C}^*$–bundle over the complex projective line $\mathbb{P}^1_\mathbb{C}$ given by the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$. So $F_{\mathbb{C}^*}$ is the complement of the zero section in the total space of $\mathcal{O}_{\mathbb{P}^1}(1)$. Let $E_{G_\mathbb{C}}$ be an algebraic principal $G_\mathbb{C}$–bundle over $\mathbb{P}^1_\mathbb{C}$, where $G_\mathbb{C}$ is a connected reductive linear algebraic group defined over $\mathbb{C}$. A theorem due to Grothendieck says that there is a homomorphism

$$\rho : \mathbb{C}^* \rightarrow G_\mathbb{C}$$

such that $E_{G_\mathbb{C}}$ is isomorphic to the principal $G_\mathbb{C}$–bundle obtained by extending the structure group of $F_{\mathbb{C}^*}$ using $\rho$ [Gr, p. 122, Théorème 1.1]. Our aim here is to investigate the corresponding set–up for the field of real numbers.

Let $X$ be a geometrically irreducible smooth projective curve, defined over the field of real numbers, satisfying the condition that

$$\text{genus}(X) := \dim H^1(X, \mathcal{O}_X) = 0.$$ 

This condition implies that either $X$ is isomorphic to $\mathbb{P}^1_\mathbb{R}$, or it is isomorphic to the nondegenerate anisotropic conic in $\mathbb{P}^2_\mathbb{R}$ defined by the polynomial $x^2 + y^2 + z^2 = 0$ (see [Ha, p. 106, Exercise 4.7(e)]). This conic clearly does not have any real points. Let $X_\mathbb{C}$ be the base change of $X$ to $\mathbb{C}$. So $X_\mathbb{C}$ is isomorphic to $\mathbb{P}^1_\mathbb{C}$.

2000 Mathematics Subject Classification. 14H60, 14L10, 14P99.

Key words and phrases. Principal bundle, real conic, reductive group.
Let $T_c$ denote the Weil restriction of the complex torus $\mathbb{C}^*$ to the subfield $\mathbb{R}$. Therefore, $T_c$ is a two-dimensional abelian reductive real linear algebraic group. The earlier defined principal $\mathbb{C}^*$–bundle $F_{\mathbb{C}^*}$ over $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$ gives a principal $T_c$–bundle $F_{T_c}$ over $X$. More precisely, the Weil restriction of $F_{\mathbb{C}^*}$ is a principal $T_c$–bundle over the Weil restriction $\hat{X}$ of $X_{\mathbb{C}}$, and $F_{T_c}$ is the pull back of this principal $T_c$–bundle by the natural embedding of $X$ in $\hat{X}$.

Let $H_0$ be the kernel of the homomorphism $T_c \longrightarrow T_c$ defined by $z \mapsto (z/\overline{z})^2$. Therefore, $H_0$ fits in a short exact sequence of groups

$$\begin{align*}
eq & G_m \longrightarrow H_0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow e,\end{align*}$$

where $G_m = \mathbb{R}^*$ is the multiplicative group, and the projection $H_0 \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is defined by $z \mapsto z/\overline{z}$.

The above defined principal $T_c$–bundle $F_{T_c}$ admits a reduction of structure group to the subgroup $H_0 \subset T_c$, and furthermore, any two reductions of structure group of $F_{T_c}$ to $H_0$ are isomorphic (see Lemma 4.6). Let (1.1) $F_{H_0} \subset F_{T_c}$ be a reduction of structure group to $H_0$.

Our main result is the following (see Theorem 4.12):

**Theorem 1.1.** Let $G$ be a connected reductive linear algebraic group defined over $\mathbb{R}$. Let $E_G$ be a principal $G$–bundle over a nondegenerate anisotropic conic $X$ in $\mathbb{P}^2_{\mathbb{R}}$. Then there is a homomorphism

$$\rho : H_0 \longrightarrow G$$

with the following property: The principal $G$–bundle $E_G$ is isomorphic to the one obtained by extending the structure group of the principal $H_0$–bundle $F_{H_0}$ in eqn. (1.1) using the homomorphism $\rho$.

We will now describe the corresponding result for principal bundles over $\mathbb{P}^1_{\mathbb{R}}$ proved here.

Let $F_{G_m}$ denote the principal $G_m$–bundle over $\mathbb{P}^1_{\mathbb{R}}$ given by the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)$. Let $F_{\mathbb{Z}/2\mathbb{Z}}$ denote the principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $\mathbb{P}^1_{\mathbb{R}}$ defined by the natural projection $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$. Therefore, the fiber product $F_{G_m} \times_X F_{\mathbb{Z}/2\mathbb{Z}}$ over $X$ is a principal $G_m \times (\mathbb{Z}/2\mathbb{Z})$–bundle.

The following theorem is proved in Theorem 4.12.

**Theorem 1.2.** Let $E_G$ be a principal $G$–bundle over $\mathbb{P}^1_{\mathbb{R}}$, where $G$ is a connected reductive linear algebraic group defined over $\mathbb{R}$. Then there is a homomorphism

$$\rho : G_m \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow G$$

such that the principal $G$–bundle $E_G$ is isomorphic to the one obtained by extending the structure group of the principal $G_m \times (\mathbb{Z}/2\mathbb{Z})$–bundle $F_{G_m} \times_X F_{\mathbb{Z}/2\mathbb{Z}}$ using $\rho$. 
If $G$ is a compact real form, then there is no nontrivial homomorphism from $\mathbb{G}_m$ to $G$. Therefore, in that case the homomorphism $\rho$ in Theorem 1.1 and the homomorphism $\rho$ in Theorem 1.2 factor through $\mathbb{Z}/2\mathbb{Z}$.

2. Semistable principal bundles

Let $X$ be a geometrically irreducible smooth projective curve defined over the field of real numbers. Let $G$ be a connected reductive linear algebraic group defined over $\mathbb{R}$. Since $G$ is connected, it follows that $G$ is actually geometrically connected. Indeed, the connected component, containing the identity element, of the complexification of $G$ is preserved by the Galois involution, thus showing that $G$ is disconnected if its complexification is so.

A principal $G$–bundle $E_G$ over $X$ will be called semistable if for each proper parabolic subgroup $P \subset G$, and for each reduction of structure group

$$E_P \subset E_G$$

of $E_G$ to $P$, the following holds: for each anti–dominant character $\chi$ of $P$ which is trivial on the center of $G$, the line bundle $E_P(\chi)$ on $X$ associated to the principal $P$–bundle $E_P$ for $\chi$ is of nonnegative degree. We recall that a character of $P$ is called anti–dominant if the associated line bundle on $G/P$ is numerically effective.

The above definition is identical to the definition in [Ra] of semistable principal bundles over a smooth complex projective curve; see [Ra, p. 131, Lemma 2.1] to compare the above definition with Definition 1.1 in [Ra, p. 129].

Remark 2.1. A couple of remarks on the above definition:

1. A principal $G$–bundle $E_G$ is semistable if and only if for each reduction $E_P$ as in eqn. (2.1), where $P$ is a maximal proper parabolic subgroup of $G$, the inequality

$$\text{degree}(\text{ad}(E_G)/\text{ad}(E_P)) \geq 0$$

holds, where $\text{ad}(E_G)$ and $\text{ad}(E_P)$ are the adjoint vector bundles of $E_G$ and $E_P$ respectively; see [Ra, p. 131, Lemma 2.1]. (The adjoint bundle of a principal $H$–bundle $E_H$ is the vector bundle associated to $E_H$ for the adjoint action of $H$ on its own Lie algebra.)

2. The group $G$ may not have any proper parabolic subgroup. For example, if $G$ is a compact real form of a complex reductive group, then $G$ does not have any proper parabolic subgroup. In such a situation, the semistability condition is automatically satisfied.

Let

$$X_\mathbb{C} = X \times_{\mathbb{R}} \mathbb{C}$$
be the base change of $X$ to the field $\mathbb{C}$. Let

$$G_{\mathbb{C}} = G \times_{\mathbb{R}} \mathbb{C}$$

be the base change to $\mathbb{C}$. Therefore, $G_{\mathbb{C}}$ is a connected reductive linear algebraic group defined over $\mathbb{C}$.

Take a principal $G$–bundle $E_G$ over $X$. The base change to $\mathbb{C}$

$$E_G^\mathbb{C} := E_G \times_{\mathbb{R}} \mathbb{C}$$

is a principal $G_{\mathbb{C}}$–bundle over $X_{\mathbb{C}}$.

**Lemma 2.2.** The principal $G$–bundle $E_G$ over $X$ is semistable if and only if the principal $G_{\mathbb{C}}$–bundle $E_G^\mathbb{C}$ over $X_{\mathbb{C}}$ is semistable.

**Proof.** First assume $E_G$ is not semistable. Let $P$ be a proper parabolic subgroup of $G$ with $E_P \subset E_G$ a reduction as in eqn. (2.1), and let $\chi$ be an anti–dominant character of $P$, such that

$$\text{degree}(E_P(\chi)) < 0,$$

where $E_P(\chi)$ is the line bundle over $X$ associated to the principal $P$–bundle $E_P$ for the character $\chi$. Let

$$E_P^\mathbb{C} := E_P \times_{\mathbb{R}} \mathbb{C} \subset E_G \times_{\mathbb{R}} \mathbb{C}$$

be the corresponding reduction of structure group of $E_G^\mathbb{C}$ to the proper parabolic subgroup $P_{\mathbb{C}} := P \times_{\mathbb{R}} \mathbb{C}$ of $G_{\mathbb{C}}$. Let $\chi_{\mathbb{C}}$ be the character of $P_{\mathbb{C}}$ given by $\chi$. The line bundle over $X_{\mathbb{C}}$ associated to the principal $P_{\mathbb{C}}$–bundle $E_P^\mathbb{C}$ for the character $\chi_{\mathbb{C}}$ will be denoted by $E_P(\chi_{\mathbb{C}})$. Since

$$\text{degree}(E_P^\mathbb{C}(\chi_{\mathbb{C}})) = \text{degree}(E_P(\chi)),$$

the pair $(E_P^\mathbb{C}, \chi_{\mathbb{C}})$ violates the semistability condition for $E_G^\mathbb{C}$. In other words, $E_G^\mathbb{C}$ is not semistable.

To prove the converse, assume that $E_G^\mathbb{C}$ is not semistable. Let

$$E_Q \subset E_G^\mathbb{C}$$

be the Harder–Narasimhan reduction of $E_G^\mathbb{C}$; here $Q \subset G_{\mathbb{C}}$ is some proper parabolic subgroup determined uniquely by $E_G^\mathbb{C}$ up to an inner automorphism. Once a subgroup $Q$ in the conjugacy class is fixed, the Harder–Narasimhan reduction $E_Q$ is uniquely determined (see [AAB, p. 694, Theorem 1] for Harder–Narasimhan reduction).

The adjoint bundle $\text{ad}(E_Q) \subset \text{ad}(E_G^\mathbb{C})$ is one of the terms in the Harder–Narasimhan filtration of $\text{ad}(E_G^\mathbb{C})$. More precisely, $\text{ad}(E_Q)$ is the term in the Harder–Narasimhan filtration of $\text{ad}(E_G^\mathbb{C})$ whose quotient by the previous term is of degree zero (see [AAB, p. 702, Lemma 6]). The Harder–Narasimhan filtration of $\text{ad}(E_G^\mathbb{C})$ is the base change to $\mathbb{C}$ of the Harder–Narasimhan filtration of $\text{ad}(E_G)$; this follows immediately from [La, p. 97, Proposition 3]. In particular, $\text{ad}(E_Q)$ is the base change to $\mathbb{C}$ of a subbundle of $\text{ad}(E_G)$.
Therefore, $Q$ can be taken to be the base change to $C$ of some parabolic subgroup of $G$. Once $Q$ is the base change to $C$ of a parabolic subgroup $P \subset G$, using the above mentioned fact that $\text{ad}(E_Q)$ is the base change to $C$ of a subbundle of $\text{ad}(E_G)$ it follows that $E_Q$ is the base change to $C$ of a reduction of structure group

\begin{align}
E_P & \subset E_G 
\end{align}

of the principal $G$–bundle $E_G$ to the subgroup $P$.

Let $g$ (respectively, $p$) be the Lie algebra of $G$ (respectively, $P$). Let $\chi$ be the character of $P$ defined by the one–dimensional $P$–module $\text{det}_g/p = \wedge^\text{top} g/p$; the action of $P$ is given by the adjoint action of $P$ on $g$. We have

$$\text{degree}(E_P(\chi)) = \text{degree}(\text{ad}(E_G)/\text{ad}(E_P)) = \text{degree}(\text{ad}(E^C_P)/\text{ad}(E_Q)) < 0.$$ 

The last inequality follows from the fact that the subbundle $\text{ad}(E_Q) \subset \text{ad}(E_G)$ is the term in the Harder–Narasimhan filtration of $\text{ad}(E_G)$ whose quotient by the previous term in the filtration has degree zero. Consequently, $E_G$ is not semistable. This completes the proof of the lemma. \hfill $\square$

A principal $G_C$–bundle $F_{G_C}$ over $X_C$ is semistable if and only if the adjoint vector bundle $\text{ad}(F_{G_C})$ is semistable [AAB, p. 698, Lemma 3]. Therefore, Lemma 2.2 has the following corollary.

**Corollary 2.3.** A principal $G$–bundle $E_G$ over $X$ is semistable if and only if the adjoint vector bundle $\text{ad}(E_G)$ is semistable.

For a parabolic subgroup $P$ of $G$, its Levi quotient will be denoted by $L(P)$. So $L(P)$ is the quotient of $P$ by its unipotent radical.

Take a principal $G$–bundle $E_G$ over $X$. Assume that $E_G$ is not semistable. The reduction $E_P \subset E_G$ in eqn. (2.5), which is obtained from $E_Q$, evidently satisfies all the conditions of a Harder–Narasimhan reduction. Therefore, we have the following corollary:

**Corollary 2.4.** Let $E_G$ be a principal $G$–bundle over $X$ which is not semistable. Then there is a proper parabolic subgroup $P \subset G$ and a reduction of structure group $E_P \subset E_G$ to $P$ satisfying the following two conditions:

1. The principal $L(P)$–bundle obtained by extending the structure group of $E_P$, using the projection of $P$ to its Levi quotient $L(P)$, is semistable.
2. For any nontrivial character $\chi$ of $P$ which is trivial on the center of $G$, and which can be expressed as a nonnegative integral combination of simple roots, the line bundle $E_P(\chi)$ over $X$, associated to the principal $P$–bundle $E_P$ for the character $\chi$ of $P$, has degree strictly greater than zero.

The subgroup $P$ is unique up to an inner automorphism, and the subbundle $\text{ad}(E_P) \subset \text{ad}(E_G)$ is canonical (it does not depend on the choice of $P$). Once $P$ is fixed in the conjugacy class, the reduction $E_P$ is uniquely determined.
Proof. Since the reduction $E_Q$ in eqn. (2.4) satisfies the two conditions in Corollary 2.4 (see [AAB, p. 712, Theorem 6]), we conclude that Corollary 2.4 holds.

The reduction $E_P$ in Corollary 2.4 will be called a \textit{Harder–Narasimhan reduction of $E_G$}. See [AAB, p. 694, Theorem 1] for an equivalent formulation of the Harder–Narasimhan reduction.

3. Reduction to torus over a nondegenerate conic

Let $X$ be a geometrically irreducible smooth projective curve of genus zero defined over $\mathbb{R}$. As mentioned in Section 1, either $X$ is isomorphic to $\mathbb{P}^1_\mathbb{R}$ or it is isomorphic to the anisotropic conic in $\mathbb{P}^2_\mathbb{R}$ defined by the homogeneous polynomial $x^2 + y^2 + z^2$. As before, let $G$ be a connected reductive linear algebraic group defined over $\mathbb{R}$.

\textbf{Proposition 3.1.} Let $E_G$ be a semistable principal $G$–bundle over $X$. Then $E_G$ admits a reduction of structure group to a maximal torus of $G$.

Proof. Let $E^C_G := E_G \times_{\mathbb{R}} \mathbb{C}$ be the principal $G_C$–bundle over $X_C$ given by $E_G$ (see eqn. (2.3)). From Lemma 2.2 it follows that $E^C_G$ is semistable. Therefore, the adjoint vector bundle $\text{ad}(E^C_G)$ is semistable [AAB, p. 698, Lemma 3].

Since $G_C$ is reductive, its Lie algebra $\mathfrak{g}_C$ as a $G_C$–module is self–dual. Therefore, the vector bundle $\text{ad}(E^C_G)$ is self–dual. In particular, we have

$$\text{degree}(\text{ad}(E^C_G)) = 0.$$ 

The complexification $X_C$ is isomorphic to $\mathbb{P}^1_\mathbb{C}$. Any algebraic vector bundle over $\mathbb{P}^1_\mathbb{C}$ decomposes into a direct sum of line bundles [Gr, p. 122, Théorème 1.1]. Since $\text{ad}(E^C_G)$ is semistable of degree zero, this implies that $\text{ad}(E^C_G)$ is a trivial vector bundle.

Set

$$\mathfrak{g}^0 := H^0(X_C, \text{ad}(E^C_G)).$$

The Lie algebra structure of the fibers of $\text{ad}(E^C_G)$ induce a Lie algebra structure on $\mathfrak{g}^0$. This Lie algebra is isomorphic to the Lie algebra $\mathfrak{g}_C$ of $G_C$ because $\text{ad}(E^C_G)$ is a trivial vector bundle with fibers isomorphic to $\mathfrak{g}_C$.

Let $\sigma$ be the self–map of $X_C$ given by the action of the nontrivial element in $\text{Gal}(\mathbb{C}/\mathbb{R})$. Therefore, $\sigma$ is an anti–holomorphic involution of $X_C$. Let

$$\tilde{\sigma} : \text{ad}(E^C_G) \longrightarrow \sigma^*\text{ad}(E^C_G)$$

be the holomorphic isomorphism of vector bundles given by the action of the nontrivial element in $\text{Gal}(\mathbb{C}/\mathbb{R})$. Here $\text{ad}(E^C_G)$ is the smooth vector bundle over $X_C$ whose fiber over any point $x \in X_C$ is identified, as a real vector space, with the fiber $\text{ad}(E^C_G)_x$, while this
identification is conjugate linear. The vector bundle $\sigma^*\text{ad}(E_G)$ has a natural holomorphic structure. We note that the composition
\[
\text{ad}(E_G) \xrightarrow{\sigma} \text{ad}(E_G) \xrightarrow{\sigma^*} \text{ad}(E_G) = \text{ad}(E_G)
\]
is the identity automorphism of $\text{ad}(E_G)$.

We have a conjugate linear involution
\[\sigma^0 : g^0 = H^0(X_C, \text{ad}(E_G)) \longrightarrow g^0\]
defined by $\alpha \mapsto -\tilde{\sigma}(\alpha)$, where $g^0$ is defined in eqn. (3.1) and $\tilde{\sigma}$ is the isomorphism in eqn. (3.2). Note that using the natural $\mathbb{R}$-linear identification of $\text{ad}(E_G)$ with $\text{ad}(E_G)$, the image $\tilde{\sigma}(\alpha)$ is a smooth section of $\text{ad}(E_G)$; this section is evidently holomorphic.

It is easy to see that $\sigma^0$ preserves the Lie algebra structure of $g^0$. The real Lie algebra given by the fixed point set
\[g^0)^{\sigma^0} \subset g^0\]
is isomorphic to the Lie algebra $g$ of $G$. Since $\sigma^0$ is a conjugate linear involution of $g^0$, the complexification $(g^0)^{\sigma^0} \otimes_{\mathbb{R}} \mathbb{C}$ is identified with $g^0$. In particular, the subset $(g^0)^{\sigma^0}$ is Zariski dense in $g^0$.

An element of $g^0$ is called semisimple if its adjoint action on $g^0$ is completely reducible. A semisimple element of $g^0$ is called regular if its centralizer is a Cartan subalgebra of $g^0$. The set of regular semisimple elements in $g^0$ is a Zariski open dense subset (see [Hu2, p. 28, Theorem 2.5]). We noted above that $(g^0)^{\sigma^0}$ in eqn. (3.4) is Zariski dense in $g^0$. Therefore, $(g^0)^{\sigma^0}$ contains some regular semisimple elements of $g^0$. Fix an element
\[\omega \in (g^0)^{\sigma^0}\]
which is a regular semisimple element of $g^0$.

Since $\text{ad}(E_G)$ is the complexification of the adjoint vector bundle $\text{ad}(E_G)$, we have a canonical identification
\[(g^0)^{\sigma^0} \longrightarrow H^0(X, \text{ad}(E_G)).\]
Let
\[\omega_\alpha \in H^0(X, \text{ad}(E_G))\]
be the section corresponding to $\omega$ in eqn. (3.5). On the other hand, the two Lie algebras $(g^0)^{\sigma^0}$ and $g$ (the Lie algebra of $G$) are identified up to an inner conjugation. Therefore, $\omega$ gives a conjugacy class in $g$. Fix an element of the Lie algebra
\[\omega_c \in g\]
in the conjugacy class given by $\omega$. Let
\[T_\omega \subset G\]
be the centralizer of $\omega_c$ for the adjoint action. Using the fact that the element $\omega \in g^0$ is regular semisimple it follows that $T_\omega$ is a maximal torus of $G$. More precisely, the
subgroup of a complex reductive group $G'$ generated by the one parameter subgroup of it defined by a fixed regular semisimple element of the Lie algebra of $G'$, together with the connected component, containing the identity element, of the center of $G'$, is a maximal torus of $G'$. Since the centralizer of maximal torus $T'$ in $G'$ is $T'$ itself [Hu1, p. 140, Corollary A], we conclude that $T_\omega$ is a maximal torus of $G$.

We recall that the adjoint vector bundle $\text{ad}(E_G)$ is a quotient of $E_G \times \mathfrak{g}$. Let $q : E_G \times \mathfrak{g} \rightarrow \text{ad}(E_G)$ be the quotient map. Let $p_1 : E_G \times \mathfrak{g} \rightarrow E_G$ be the projection to the first factor.

Finally, consider the projection

$$E_{T_\omega} := p_1((E_G \times \{\omega_c\}) \cap q^{-1}(\omega_a(X))) \subset E_G,$$

where $\omega_a : X \rightarrow \text{ad}(E_G)$ is the section in eqn. (3.6), and $\omega_c$ is the element in eqn. (3.7). It is straightforward to check that $E_{T_\omega}$ is a reduction of structure group of the principal $G$–bundle $E_G$ to the subgroup $T_\omega$ in eqn. (3.8). This completes the proof of the proposition.

Let $P$ be a parabolic subgroup of a connected reductive linear algebraic group $G$ defined over $\mathbb{R}$. Consider the short exact sequence of groups

$$e \rightarrow R_u(P) \rightarrow P \rightarrow L(P) \rightarrow e,$$

where $R_u(P)$ is the unipotent radical of $P$, and $L(P)$ is the Levi quotient of $P$. As a special case of a result of Mostow, this exact sequence is right split (see [Bo, p. 158, § 11.22]). In other words, $P$ is a semidirect product $R_u(P) \rtimes L(P)$ of the unipotent radical $R_u(P)$ and the Levi quotient $L(P)$. Hence there is a subgroup of $P$ that projects isomorphically onto $L(P)$. The result of Mostow also says that any two such subgroups of $P$ are conjugate [Bo, p. 158, § 11.22]. Fix a subgroup of $P$ that projects isomorphically onto $L(P)$. This subgroup will also be denoted by $L(P)$; this subgroup of $P$ will be called the Levi subgroup of $P$.

**Proposition 3.2.** Let $E_P$ be a principal $P$–bundle over the curve $X$ of genus zero that satisfies the following condition: There is a principal $G$–bundle over $X$ whose Harder–Narasimhan reduction, described in Corollary 2.4, is $E_P$. Then $E_P$ admits a reduction of structure group to the Levi subgroup $L(P) \subset P$.

**Proof.** Let $\text{Ad}(E_P) := E_P \times_P P$ denote the adjoint bundle of $E_P$. We recall that $\text{Ad}(E_P)$ is the fiber bundle over $X$ associated to the principal $P$–bundle $E_P$ for the adjoint action of $P$ on itself. Therefore, $\text{Ad}(E_P)$ is a quotient of $E_P \times P$, and it is a group–scheme over $X$. The Lie algebra bundle for $\text{Ad}(E_P)$ is the adjoint vector bundle $\text{ad}(E_P)$.
Let
\[(3.10) \quad E_P(R_u(P)) := E_P \times_P R_u(P) \subset E_P \times_P \mathbb{P} =: \text{Ad}(E_P)\]
be the subgroup–scheme over \(X\) given by the \(P\) invariant subgroup \(R_u(P) \subset P\). We note that \(E_P(R_u(P))\) is associated to the principal \(P\)–bundle \(E_P\) for the adjoint action of \(P\) on \(R_u(P)\).

The group–scheme \(\text{Ad}(E_P)\) has a natural action on the principal \(P\)–bundle \(E_P\). Indeed, the map
\[(E_P \times P) \times E_P \longrightarrow P\]
defined by \((z, p, zp') \longmapsto zpp'\), where \(z \in E_P\) and \(p, p' \in P\), descends to a map
\[((E_P \times P)/P) \times E_P = \text{Ad}(E_P) \times E_P \longrightarrow P\]
giving the action of \(\text{Ad}(E_P)\) on \(E_P\). This action of \(\text{Ad}(E_P)\) on \(E_P\) clearly commutes with the right action of \(P\) on \(E_P\).

The above action of \(\text{Ad}(E_P)\) on \(E_P\) gives an action of \(\text{Ad}(E_P)\) on any fiber bundle over \(X\) associated to \(E_P\). To see this induced action, let \(Y\) be a variety on which \(P\) acts on the left, and let
\[E_P(Y) := E_P \times_P Y\]
be the fiber bundle over \(X\) associated to \(E_P\) for its action on \(Y\). Then the map
\[\text{Ad}(E_P) \times (E_P \times Y) \longrightarrow E_P \times Y\]
defined by \((g, (z, y)) \longmapsto (gz, y)\) descends to an action of \(\text{Ad}(E_P)\) on the quotient space \(E_P(Y)\) of \(E_P \times Y\).

Consider the quotient space
\[E_P(P/L(P)) := E_P/L(P)\]
which is a fiber bundle over \(X\) with \(P/L(P)\) as the fiber; here \(L(P)\) is the Levi subgroup of \(P\). We note that \(E_P(P/L(P))\) is the fiber bundle associated to \(E_P\) for the left–translation action of \(P\) on \(P/L(P)\). Restricting the action of \(\text{Ad}(E_G)\) on \(E_P(P/L(P))\) to the subgroup–scheme \(E_P(R_u(P))\) (defined in eqn. (3.10)) we get an action of \(E_P(R_u(P))\) on \(E_P(P/L(P))\). Using the fact that the subgroup \(L(P)\) of \(P\) projects isomorphically onto the quotient \(P/R_u(P)\) (see eqn. (3.9)) it follows that \(E_P(P/L(P))\) is a torsor for the group–scheme \(E_P(R_u(P))\). In other words, the group–scheme \(E_P(R_u(P))\) acts freely transitively on the fiber bundle \(E_P(P/L(P))\).

Isomorphism classes of torsors for \(E_P(R_u(P))\) are parametrized by \(H^1(X, E_P(R_u(P)))\). Since \(E_P\) is Zariski locally trivial, we may use Zariski topology. Let
\[(3.11) \quad \theta \in H^1(X, E_P(R_u(P)))\]
be the element corresponding to the above torsor \(E_P(P/L(P))\).

The Lie algebra of the unipotent radical \(R_u(P)\) of \(P\) will be denoted by \(R_u(\mathfrak{p})\). So \(R_u(\mathfrak{p})\) is the nilpotent radical of the Lie algebra of \(P\). We note that the adjoint action \(P\)
on its own Lie algebra leaves the subalgebra $R_n(p)$ invariant, just as the adjoint action of $P$ on itself leaves the subgroup $R_u(P)$ invariant. Let $E_P(R_n(p))$ be the vector bundle over $X$ associated to the principal $P$–bundle $E_P$ for the $P$–module $R_n(p)$. Since the adjoint action of $P$ on $R_n(p)$ preserves its Lie algebra structure, it follows that $E_P(R_n(p))$ is a bundle of Lie algebras over $X$. The Lie algebra bundle for the group–scheme $E_P(R_u(P))$ is evidently identified with $E_P(R_n(p))$.

Let

$$0 = E_{-\ell-1} \subset E_{-\ell} \subset E_{-\ell+1} \subset \cdots \subset E_{-2} \subset E_{-1} = E_P(R_n(p))$$

be the Harder–Narasimhan filtration of the vector bundle. We know that

$$[E_{-j}, E_{-1}] \subset E_{-j-1}$$

for all $j \in [1, \ell]$; see [AAB, p. 699, (2)]. From eqn. (3.13) we conclude the following:

- Each $E_{-j}$ is a bundle of ideals in the Lie algebra bundle $E_P(R_u(P))$.
- For each $j \in [1, \ell]$, the quotient Lie algebra bundle $E_{-j}/E_{-j-1}$ is abelian.

Given a unipotent linear algebraic group $U$, if its Lie algebra $u$ is abelian, then the exponential map

$$\exp : u \rightarrow U$$

is an isomorphism of algebraic groups. Hence the filtration in eqn. (3.12) gives a filtration of normal subgroup-schemes of $E_P(R_u(P))$

$$e_X = G_{-\ell-1} \subset G_{-\ell} \subset G_{-\ell+1} \subset \cdots \subset G_{-2} \subset G_{-1} = E_P(R_u(P))$$

such that

$$G_{-j}/G_{-j-1} = E_{-j}/E_{-j-1}$$

as group–schemes for all $j \in [1, \ell]$; here $e_X$ is the group–scheme over $X$ for the trivial group.

For each $j \in [1, \ell]$, the quotient $E_{-j}/E_{-j-1}$ is a semistable vector bundle over $X$ of positive degree. (We recall that $\text{ad}(E_P)$ is the term in the Harder–Narasimhan filtration of $\text{ad}(E_G)$ whose quotient by the previous term is of degree zero.) Since $(E_{-j}/E_{-j-1})^* \otimes K_X$ is a semistable vector bundle of negative degree, where $K_X$ is the canonical line bundle of $X$ (its degree is $-2$), using Serre duality we have

$$H^1(X, E_{-j}/E_{-j-1}) = H^0(X, (E_{-j}/E_{-j-1})^* \otimes K_X)^* = 0.$$  

Therefore, using eqn. (3.15) we have $H^1(X, G_{-j}/G_{-j-1}) = 0$ for all $j \in [1, \ell]$. Now using the filtration in eqn. (3.14) it follows that

$$H^1(X, E_P(R_u(P))) = 0.$$  

To prove this note that since $H^1(X, G_{-j}/G_{-j-1}) = 0$, in order to show that

$$H^1(X, G_{-j}) = 0$$
it suffices to prove that $H^1(X, G_{j-1}) = 0$. In particular, the element $\theta$ in eqn. (3.11) vanishes. Hence $E_P(P/L(P)) = E_P/L(P)$ is a trivial $E_P(R_u(P))$-torsor. In particular, the fiber bundle $E_P/L(P)$ over $X$ admits a section.

Any section $\sigma : X \to E_P/L(P)$ of the fiber bundle $E_P/L(P)$ gives a reduction of structure group of $E_P$ to $L(P)$. Indeed, the inverse image of the subvariety $\sigma(X) \subset E_P/L(P)$ for the quotient map $E_P \to E_P/L(P)$ is a reduction of structure group of $E_P$ to $L(P)$. This completes the proof of the proposition. \hfill \Box

Proposition 3.1 and Proposition 3.2 together have the following corollary.

**Corollary 3.3.** Let $E_G$ be a principal $G$–bundle over the curve $X$ of genus zero, where $G$ is a connected reductive linear algebraic group over $\mathbb{R}$. Then $E_G$ admits a reduction of structure group to a maximal torus of $G$.

**Proof.** If $E_G$ is semistable, then it follows from Proposition 3.1.

Now assume that $E_G$ is not semistable. Let $E_P$ be the Harder–Narasimhan reduction of $E_G$ (see Corollary 2.4). From Proposition 3.2 we know that $E_P$ admits a reduction of structure group $E_{L(P)} \subset E_P$ to the Levi subgroup $L(P)$ of $P$. Hence $E_G$ admits a reduction of structure group to $L(P)$. The principal $L(P)$–bundle obtained by extending the structure group of $E_P$ using the projection of $P$ to $L(P) = P/R_u(P)$ is semistable (see Corollary 2.4). Since the principal $L(P)$–bundle $E_{L(P)}$ is identified with this principal $L(P)$–bundle obtained by extending the structure group of $E_P$, we conclude that $E_{L(P)}$ is semistable.

Therefore, using Proposition 3.1 we now conclude that the principal $G$–bundle $E_G$ admits a reduction of structure group to a maximal torus of the Levi subgroup $L(P)$. A maximal torus of $L(P)$ is also a maximal torus of $G$. This completes the proof of the corollary. \hfill \Box

Consider the base change $X_\mathbb{C}$ in eqn. (2.2) of $X$ to $\mathbb{C}$. Therefore, $X_\mathbb{C}$ is isomorphic to $\mathbb{P}^1_\mathbb{C}$.

**Definition 3.4.** The natural projection

$$f : X_\mathbb{C} \to X$$

defines a principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $X$. This principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $X$ will be denoted by $F_{\mathbb{Z}/2\mathbb{Z}}$.

We note that $F_{\mathbb{Z}/2\mathbb{Z}}$ is a nontrivial principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $X$. To see this observe that $X_\mathbb{C}$ is irreducible.

**Lemma 3.5.** Let $\Gamma$ be a finite group and $E_\Gamma$ a principal $\Gamma$–bundle over the real projective curve $X$ of genus zero. Then there is a homomorphism

$$\rho : \mathbb{Z}/2\mathbb{Z} \to \Gamma$$
such that the principal $\Gamma$–bundle $E_\Gamma$ is isomorphic to the one obtained by extending the structure group of the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ in Definition 3.4 using $\rho$.

Proof. Consider the short exact sequence of étale fundamental groups

$$e \rightarrow \pi_1(X_\mathbb{C}, x) \rightarrow \pi_1(X, x) \rightarrow \text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \rightarrow e,$$

where $x$ is a point of $X_\mathbb{C}$ (see [Mu, p. 153, Theorem (8.1.1)]). Using it and the fact that $\pi_1(X_\mathbb{C}, x) = e$ it follows that $\pi_1(X, x) = \mathbb{Z}/2\mathbb{Z}$. We noted earlier that the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ over $X$ in Definition 3.4 is nontrivial. Therefore, all principal bundles over $X$ with a finite group as the structure group are obtained from $F_{\mathbb{Z}_2}$ by extension of structure group. This completes the proof of the lemma. □

4. Reduction of a torus bundle over a conic

Let $T_s = \mathbb{G}_m$ be the split torus of dimension one defined over $\mathbb{R}$. The anisotropic torus of dimension one defined over $\mathbb{R}$ will be denoted by $T_a$ (see [Bo, p. 121, § 8.16]). So the base change of $T_a$ to $\mathbb{C}$ is isomorphic to $\mathbb{C}^*$, and the action of the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^*$ is the involution defined by

$$(4.1) \quad z \mapsto \frac{1}{z}.$$ 

Let $T_c$ denote the Weil restriction of the complex torus $\mathbb{C}^*$ to the subfield of real numbers. Therefore, $T_c$ is a quotient of $T_s \times T_a$ by $\mathbb{Z}/2\mathbb{Z}$. It can be shown that $T_c \neq T_s \times T_a$.

To prove this we note that any torus $T'$ has a unique maximal anisotropic subtorus $T'_a$ and a unique maximal split subtorus $T'_d$; the two subgroups $T'_a$ and $T'_d$ generate $T'$, while $T'_a \cap T'_d$ is a finite group [Bo, p. 121, Proposition]. In the case of $T' = T_c$, we have $T'_a \cap T'_d = \mathbb{Z}/2\mathbb{Z}$. Hence $T_c \neq T_s \times T_a$.

The following lemma is well known (see [PR, p. 76, line 8] for proof):

Lemma 4.1. Let $T$ be an algebraic torus defined over $\mathbb{R}$. Then $T$ is isomorphic to the Cartesian product $(T_s)^{n_1} \times (T_a)^{n_2} \times (T_c)^{n_3}$ for some nonnegative integers $n_1$, $n_2$ and $n_3$.

As in Section 3, we will denote by $X$ a geometrically irreducible smooth projective curve of genus zero defined over $\mathbb{R}$. We will construct a nontrivial principal $T_a$–bundle over $X$.

Let

$$(4.2) \quad h_0 : \mathbb{Z}/2\mathbb{Z} \rightarrow T_a$$

be the injective homomorphism whose image is $\{\pm 1\}$. Let $F_{T_a}$ be the principal $T_a$–bundle over $X$ obtained by extending the structure group of the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ using the homomorphism $h_0$ in eqn. (4.2); see Definition 3.4 for $F_{\mathbb{Z}_2}$.

Proposition 4.2. The above defined principal $T_a$–bundle $F_{T_a}$ over the genus zero curve $X$ is nontrivial. Any nontrivial principal $T_a$–bundle over $X$ is isomorphic to $F_{T_a}$. 

Proof. Take any principal $T_a$–bundle $E_{T_a}$ over $X$. Let
\[ E_{T_a}^\mathbb{C} := E_{T_a} \times_{\mathbb{R}} \mathbb{C} \]
be the base change to $\mathbb{C}$. Therefore, $E_{T_a}^\mathbb{C}$ is a principal
bundle over $X_\mathbb{C} = X \times_{\mathbb{R}} \mathbb{C}$ (see eqn. (2.2)). Let
\[ (4.3) \quad \sigma : X_\mathbb{C} \longrightarrow X_\mathbb{C} \]
be the action of the nontrivial element in $\text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. Therefore, $\sigma$ is an anti-
holomorphic involution of $X_\mathbb{C} \cong \mathbb{P}_\mathbb{C}^1$.

Let $L$ denote the algebraic line bundle over $X_\mathbb{C}$ defined by the principal $\mathbb{C}^*$–bundle $E_{T_a}^\mathbb{C}$. The dual line bundle of $L$ will be denoted by $L^*$. Let $L^\mathbb{C}$ be the real analytic complex line bundle over $X_\mathbb{C}$ whose underlying real vector bundle of rank two is identified with that of $L^*$, while the complex structure of the fibers of $L^\mathbb{C}$ are conjugate to the complex structure of the fibers of $L$. In other words, the identification of $L^*$ with $L^\mathbb{C}$ is fiberwise conjugate linear. The pull back $\sigma^* L^\mathbb{C}$ has a natural structure of a complex algebraic line bundle over $X_\mathbb{C}$. Since $L$ is given by the complexification of a principal $T_a$–bundle over $X$, there is a holomorphic isomorphism of line bundles
\[ (4.4) \quad \eta : L \longrightarrow \sigma^* L^\mathbb{C} \]
satisfying a condition which we will describe. Let
\[ \sigma^* \eta^* : \sigma^* L^\mathbb{C} \longrightarrow \sigma^* (\sigma^* L^\mathbb{C})^* = L \]
be the holomorphic isomorphism given by $\eta$. The condition on $\eta$ says that the composition
\[ (4.5) \quad \sigma^* \eta^* \circ \eta = \text{Id}_L. \]
This condition follows from the fact that the Galois involution of the complexification $(T_a)_\mathbb{C}$ is the one in eqn. (4.1).

Conversely, any pair $(L, \eta)$, where
\begin{itemize}
  \item $L$ is a holomorphic line bundle over $X_\mathbb{C}$, and
  \item $\eta : L \longrightarrow \sigma^* L^\mathbb{C}$ is a holomorphic isomorphism of line bundles satisfying the identity in eqn. (4.5),
\end{itemize}
define a principal $T_a$–bundle over $X$. Furthermore, isomorphisms between two principal $T_a$–bundles over $X$ are parametrized by isomorphisms between the corresponding pairs.

Take any pair $(L, \eta)$ satisfying the above conditions. For a topological complex vector bundle $V$ over $X_\mathbb{C}$ we have
\[ \text{degree}(V^*) = \text{degree}(\sigma^* V) = \text{degree}(V) \quad \text{and} \quad \text{degree}(V^*) = -\text{degree}(V). \]
Hence $\text{degree}(\sigma^* L^\mathbb{C}) = -\text{degree}(L)$. Therefore, the existence of the isomorphism $\eta : L \longrightarrow \sigma^* L^\mathbb{C}$ implies that $\text{degree}(L) = 0$. Thus $L$ is a trivial line bundle over $X_\mathbb{C} \cong \mathbb{P}_\mathbb{C}^1$. 

Fix a trivialization of $L$. This trivialization gives a trivialization of the dual line bundle $L^*$, hence we also have a trivialization of $\sigma^*L^*$. Using these trivializations of $L$ and $\sigma^*L^*$, the isomorphism $\eta$ corresponds to multiplication by some nonzero complex number $\lambda \in \mathbb{C}^*$. It is straightforward to check that the isomorphism $\sigma^*\eta^*$ in eqn. (4.5) is given by multiplication with $1/\lambda$ with respect to the trivializations. Therefore, from the given condition that $\eta$ satisfies the identity in eqn. (4.5) we conclude that $\lambda \in \mathbb{R}^*$. By altering trivialization we see that $\lambda = \pm 1$.

The case of $\lambda = 1$ corresponds to the trivial principal $T_a$-bundle over $X$. Therefore, there is at most one nontrivial principal $T_a$-bundle over $X$ up to an isomorphism.

Consider the short exact sequence of algebraic groups

$$e \longrightarrow \mathbb{Z}/2\mathbb{Z} = \pm 1 \hookrightarrow T_a \xrightarrow{\psi} T_a \longrightarrow e,$$

where $\psi$ is defined by $g \longmapsto g^2$. Take any principal $T_a$-bundle $E_{T_a}$ over $X$. Let $E'_{T_a}$ be the principal $T_a$-bundle over $X$ obtained by extending the structure group of $E_{T_a}$ using the homomorphism $\psi$ in eqn. (4.6).

From our earlier observation that $E_{T_a}$ corresponds to either $\lambda = 1$ or $\lambda = -1$ it follows immediately that $E'_{T_a}$ corresponds to $1$. Hence $E'_{T_a}$ is a trivial principal $T_a$-bundle. Now using the short exact sequence in eqn. (4.6) we conclude that isomorphism classes of principal $T_a$-bundles over $X$ are parametrized by isomorphism classes of principal $\mathbb{Z}/2\mathbb{Z}$-bundles over $X$; a principal $\mathbb{Z}/2\mathbb{Z}$-bundle gives a principal $T_a$-bundle by extending the structure group using the inclusion homomorphism in eqn. (4.6).

Since the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $F_{\mathbb{Z}/2}$ in Definition 3.4 is nontrivial, we conclude that the principal $T_a$-bundle $F_{T_a}$ over $X$ given by it using the homomorphism $h_0$ is nontrivial. This completes the proof of the proposition. \qed

**Remark 4.3.** A couple of remarks on the proof of Proposition 4.2:

- Let $L$ denote the trivial complex line bundle over $X_C$ equipped with a trivialization. So $\sigma^*L^*$ is equipped with an induced trivialization. As we saw in the proof of Proposition 4.2, any isomorphism as in eqn. (4.4) satisfying the identity in eqn. (4.5) corresponds to multiplication by some $\lambda \in \{\pm 1\}$ with respect to the trivializations of $L$ and $\sigma^*L^*$. It is easy to see directly that the two pairs $(O_{X_C}, 1)$ and $(O_{X_C}, -1)$ are not isomorphic. Indeed, finding an isomorphism between them would amount to finding a $\mu \in \mathbb{C}^*$ such that $\mu = -1/\overline{\mu}$, which is impossible.

- It may also be pointed out that real algebraic line bundles over $X$ are in natural bijective correspondence with pairs of the form $(\zeta, \beta)$, where $\zeta$ is a holomorphic line bundle over $X_C$, and

$$\beta : \zeta \longrightarrow \sigma^*\overline{\zeta}$$

is a holomorphic isomorphism of line bundles, such that the composition

$$\zeta \xrightarrow{\beta} \sigma^*\overline{\zeta} \xrightarrow{\sigma^*(\overline{\sigma^*\zeta})} \sigma^*(\sigma^*(\sigma^*\zeta)) = \zeta$$
Proposition 4.2 has the following corollary:

**Corollary 4.4.** Any principal $T_a$–bundle over $X$ admits a reduction of structure group to the subgroup $\{\pm 1\} \subset T_a$.

Consider the split torus $T_s = \mathbb{G}_m$. Principal $T_s$–bundles over $X$ are in bijective correspondence with the real algebraic line bundles over $X$. Given a principal $T_s$–bundle, the corresponding line bundle is the one associated to it by the standard action of $\mathbb{G}_m$ on $\mathbb{R}$. Conversely, given a line bundle $\xi$ over $X$, the complement of the zero section in the total space of $\xi$ is a principal $T_s$–bundle.

The following proposition is easy to prove (it follows from [BN, p. 1208, Theorem 1.1] in the case of anisotropic conic, and it follows from [BN, p. 1210, Proposition 3.1] in the case of projective line).

**Proposition 4.5.** Assume that $X$ is isomorphic to a nondegenerate anisotropic conic in $\mathbb{P}^2_\mathbb{R}$. The group of line bundles over $X$ is isomorphic to $\mathbb{Z}$, and it is generated by the tangent bundle $TX$.

The group of line bundles over $\mathbb{P}^1_\mathbb{R}$ is isomorphic to $\mathbb{Z}$, and it is generated by the tautological line bundle $O_{\mathbb{P}^1_\mathbb{R}}(1)$.

We will now consider the principal $T_c$–bundles over $X$, where $T_c$ is the two–dimensional torus defined earlier (see Lemma 4.1). First observe that there is a short exact sequence of algebraic groups

\begin{equation}
  e \longrightarrow T_s \hookrightarrow T_c \xrightarrow{\phi} T_a \longrightarrow e,
\end{equation}

where the injective homomorphism is given by the inclusion of $\mathbb{R}^*$ in $\mathbb{C}^*$, and the projection $\phi$ is defined by $z \mapsto z/z$. Consider the subgroup

\begin{equation}
  \mathbb{Z}/2\mathbb{Z} = \{\pm 1\} \subset T_a.
\end{equation}

Let

\begin{equation}
  H_0 := \phi^{-1}(\mathbb{Z}/2\mathbb{Z}) \subset T_c
\end{equation}

be the closed subgroup, where $\phi$ is the projection in eqn. (4.7). Therefore, from eqn. (4.7) we have the exact sequence

\begin{equation}
  e \longrightarrow T_s \hookrightarrow H_0 \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \longrightarrow e.
\end{equation}

**Lemma 4.6.** Any principal $T_c$–bundle over the genus zero curve $X$ admits a reduction of structure group to the subgroup $H_0$ defined in eqn. (4.9).

Let $E_{T_c}$ be a principal $T_c$–bundle over $X$, and let $E_{H_0}^1 \subset E_{T_c}$ and $E_{H_0}^2 \subset E_{T_c}$ be reductions of structure group of $E_{T_c}$ to the subgroup $H_0$. Then the two principal $H_0$–bundles $E_{H_0}^1$ and $E_{H_0}^2$ over $X$ are isomorphic.
Proof. Let $E_{T_c}$ be a principal $T_c$–bundle over $X$. Let $E_{T_a}$ be the principal $T_a$–bundle obtained by extending the structure group of $E_{T_c}$ using the projection $\phi$ in eqn. (4.7). From Corollary 4.4 we know that $E_{T_a}$ admits a reduction of structure group to the subgroup $\mathbb{Z}/2\mathbb{Z}$ in eqn. (4.8).

Let $E_{\mathbb{Z}/2\mathbb{Z}} \subset E_{T_a}$ be a reduction of structure group of $E_{T_a}$ to $\mathbb{Z}/2\mathbb{Z}$. Now the inverse image

$$q^{-1}(E_{\mathbb{Z}/2\mathbb{Z}}) \subset E_{T_c},$$

where $q : E_{T_c} \rightarrow E_{T_a}$ is the reduction of structure group of $E_{T_a}$ to the subgroup $H_0 \subset T_c$.

We will now show that any two reductions of structure group of $E_{T_c}$ to $H_0$ are isomorphic.

Let $E^{1}_{H_0}$ and $E^{2}_{H_0}$ be two reductions of structure group of $E_{T_c}$ to $H_0$. For $i = 1, 2$, let $E^{i}_{\mathbb{Z}/2\mathbb{Z}}$ be the principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $X$ obtained from $E^{i}_{H_0}$ by extending the structure group using the projection $\phi$ in eqn. (4.10). Therefore, $E^{i}_{\mathbb{Z}/2\mathbb{Z}}$ is a reduction of structure group of the earlier defined principal $T_a$–bundle $E_{T_a}$ to the subgroup $\mathbb{Z}/2\mathbb{Z}$.

To prove that $E^{1}_{H_0}$ and $E^{2}_{H_0}$ are isomorphic it suffice to produce a real point $t \in T_a$ such that

$$E^{2}_{\mathbb{Z}/2\mathbb{Z}} = E^{1}_{\mathbb{Z}/2\mathbb{Z}}t \subset E_{T_a}.$$  

Indeed, a lift $\tilde{t} \in \phi^{-1}(t) \subset T_c$ of any $t$ satisfying this condition, where $\phi$ is the projection in eqn. (4.7), has the property that

$$E^{2}_{H_0} = E^{1}_{H_0}\tilde{t} \subset E_{T_c},$$

which means that the automorphism of the principal $T_c$–bundle $E_{T_c}$ defined by multiplication with $\tilde{t}$ is an isomorphism of $E^{2}_{H_0}$ with $E^{1}_{H_0}$. We note that the projection $\phi$ in eqn. (4.7) is surjective on the real points; therefore, any real point $t \in T_a$ can be lifted to a real point of $T_c$.

The principal $T_a$–bundle obtained by extending the structure group of $E_{T_a}$ using the homomorphism $\psi$ in eqn. (4.6) is trivial. Therefore, any reduction of structure group of the principal $T_a$–bundle $E_{T_a}$ to the subgroup $\mathbb{Z}/2\mathbb{Z}$ is given by a morphism from $X$ to the quotient space $T_a/(\mathbb{Z}/2\mathbb{Z}) \cong T_a$. We note that there are no nonconstant maps from the irreducible projective variety $X$ to the affine variety $T_a/(\mathbb{Z}/2\mathbb{Z})$.

Since both $E^{1}_{\mathbb{Z}/2\mathbb{Z}}$ and $E^{2}_{\mathbb{Z}/2\mathbb{Z}}$ are reductions of structure group of $E_{T_a}$ to $\mathbb{Z}/2\mathbb{Z}$, there is a real point $t_0 \in T_a/(\mathbb{Z}/2\mathbb{Z})$ such that

$$z_2t_0 = z_1,$$

where $z_i \in T_a/(\mathbb{Z}/2\mathbb{Z})$, $i = 1, 2$, is the image of the constant function

$$X \rightarrow T_a/(\mathbb{Z}/2\mathbb{Z})$$

that gives the reduction $E^{i}_{\mathbb{Z}/2\mathbb{Z}}$. We note that the homomorphism $\psi$ in eqn. (4.6) is surjective on the real points. Therefore, there is a real point $t \in T_a$ such that $\psi(t) = t_0$. 

It is now straight–forward to check that this $t$ satisfies the condition in eqn. (4.11). This completes the proof of the lemma. \hfill $\Box$

**Proposition 4.7.** Let $E_{T_c}$ be a principal $T_c$–bundle over the real projective line $\mathbb{P}_R^1$. Then there is a principal $T_s$–bundle $E_{T_s}$ over $\mathbb{P}_R^1$ such that $E_{T_c}$ is obtained from $E_{T_s}$ by extending the structure group using the inclusion homomorphism in eqn. (4.7).

**Proof.** Using Lemma 4.6 we know that there is a reduction of structure group
\begin{equation}
E_{H_0} \subset E_{T_c}
\end{equation}
to the subgroup $H_0$ defined in eqn. (4.9).

Let
\begin{equation}
\tilde{\varphi} : T_c \longrightarrow \text{GL}(2, \mathbb{R})
\end{equation}
be the embedding defined by
\[a + \sqrt{-1} b \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} .\]
Therefore, $\tilde{\varphi}$ corresponds to multiplication of $\mathbb{C} = \mathbb{R}^2$ by $\mathbb{C}^*$. The restriction of $\tilde{\varphi}$ to the subgroup $H_0 \subset T_c$ will be denoted by $\varphi$.

Consider the adjoint action of $H_0$ on the Lie algebra $M(2, \mathbb{R})$ of $\text{GL}(2, \mathbb{R})$ constructed using $\varphi$. The $H_0$–module $M(2, \mathbb{R})$ decomposes into a direct sum of one–dimensional $H_0$–modules. Indeed, the four matrices
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
generate the four one–dimensional $H_0$–modules. The adjoint action of $H_0$ on $M(2, \mathbb{R})$ factors through the quotient $\mathbb{Z}/2\mathbb{Z}$ in eqn. (4.10) because the subgroup $T_s \subset H_0$ acts trivially on $M(2, \mathbb{R})$. Therefore, for any one–dimensional submodule $L$ of $M(2, \mathbb{R})$, the action of $H_0$ on $L \otimes L$ is trivial.

Consider the action of $H_0$ on $\mathbb{R}^2$ constructed using the homomorphism $\varphi$ and the standard representation of $\text{GL}(2, \mathbb{R})$. Let $V$ denote the vector bundle over $\mathbb{P}_R^1$ associated to the principal $H_0$–bundle $E_{H_0}$ in eqn. (4.12) for this $H_0$–module $\mathbb{R}^2$. The vector bundle $\text{End}(V)$ is identified with the vector bundle associated to the principal $H_0$–bundle $E_{H_0}$ for the $H_0$–module $M(2, \mathbb{R})$. Since $M(2, \mathbb{R})$ decomposes into a direct sum of one–dimensional $H_0$–modules of order two, it follows that the vector bundle $\text{End}(V)$ decomposes into a direct sum of line bundles of order two. Now we conclude that $\text{End}(V)$ is a trivial vector bundle because any line bundle over $\mathbb{P}_R^1$ of order two is trivial (see Proposition 4.5).

Since $\text{End}(V)$ is a trivial vector bundle, using the second part of Proposition 4.5 it follows that
\[V \cong \mathcal{O}_{\mathbb{P}_R^1}(\ell)^{\oplus 2}\]
for some $\ell \in \mathbb{Z}$. 

Let $F_{T_s}$ be the principal $T_s$–bundle over $\mathbb{P}^1_\mathbb{R}$ given by the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_\mathbb{R}}(1)$. Let $F_{H_0}^\ell$ be the principal $H_0$–bundle over $\mathbb{P}^1_\mathbb{R}$ obtained by extending the structure group of $F_{T_s}$ using the homomorphism

$$T_s \rightarrow T_a \subset H_0$$

defined by $g \mapsto g^\ell$ (see eqn. (4.10)). The proof of the proposition will be completed by showing that the two principal $H_0$–bundles $E_{H_0}$ and $F_{H_0}^\ell$ are isomorphic.

For that purpose, consider the short exact sequence of algebraic groups

(4.14)

$$e \rightarrow \mathbb{Z}/2\mathbb{Z} = \{\pm 1\} \rightarrow H_0 \xrightarrow{\nu} T_s \rightarrow e,$$

where $\nu$ is the norm map defined by $z \mapsto z\overline{z}$. Both the principal $H_0$–bundles $E_{H_0}$ and $F_{H_0}^\ell$ have the property that the principal $T_s$–bundle obtained by extending the structure group using the homomorphism $\nu$ in eqn. (4.14) coincides with the principal $T_s$–bundle over $\mathbb{P}^1_\mathbb{R}$ given by the line bundle $\mathcal{O}_{\mathbb{P}^1_\mathbb{R}}(2\ell)$. Therefore, to prove that $E_{H_0}$ and $F_{H_0}^\ell$ are isomorphic it suffices to show that for any principal $\mathbb{Z}/2\mathbb{Z}$–bundle $E'_\mathbb{Z}$ over $\mathbb{P}^1_\mathbb{R}$, the principal $H_0$–bundle $E'_{H_0}$ obtained by extending its structure group using the homomorphism in eqn. (4.14) is trivial.

The inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow H_0$ in eqn. (4.14) factors through the subgroup $T_s \subset H_0$ in eqn. (4.10). Therefore, to prove that $E'_{H_0}$ is a trivial it is enough to show that that the principal $T_s$–bundle $E'_T$ obtained by extending the structure group of the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $E'_\mathbb{Z}$ is trivial.

Any finite order line bundle over $\mathbb{P}^1_\mathbb{R}$ is trivial (see Proposition 4.5). In particular, the order two line bundle given by the principal $T_s$–bundle $E'_T$ is trivial. Hence the principal $H_0$–bundle $E'_{H_0}$ is trivial. This completes the proof of the proposition. \qed

Let $X$ be a nondegenerate anisotropic conic in $\mathbb{P}^2_\mathbb{R}$. Before classifying principal $T_c$–bundles over $X$, we will construct a certain principal $T_c$–bundles over $X$.

Consider the ample generator of $\text{Pic}(X_C)$ (see eqn. (2.2)). This is the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_\mathbb{C}}(1)$ over $X_C \cong \mathbb{P}^1_\mathbb{C}$. Let $E^0_{C^*}$ be the principal $\mathbb{C}^*$–bundle over $X_C$ given by this line bundle. The Weil restriction of $E^0_{C^*}$ to the subfield $\mathbb{R} \subset \mathbb{C}$ is a principal $T_c$–bundle over the Weil restriction $\widetilde{X}$ of $X_C$. (Since $T_c$ is the Weil restriction of $\mathbb{C}^*$, it follows immediately that the Weil restriction of $E^0_{C^*}$ is a principal $T_c$–bundle.) This principal $T_c$–bundle over $\widetilde{X}$ will be denoted by $E^0_{T_c}$.

**Definition 4.8.** Let $F_{T_c}$ be the principal $T_c$–bundle over $X$ obtained by pulling back the above principal $T_c$–bundle $E^0_{T_c}$ over the Weil restriction $\widetilde{X}$ using the canonical inclusion map

(4.15)\n
$$X \hookrightarrow \widetilde{X}.$$\n
Fix a reduction of structure group $F_{H_0} \subset F_{T_c}$.
to the subgroup $H_0$ of $T_c$. (From Lemma 4.6 we know that such a reduction of structure group exists, and any two reductions are isomorphic.)

**Proposition 4.9.** Any real algebraic line bundle over the nondegenerate anisotropic conic $X$ is obtained by extending the structure group of the principal $H_0$–bundle $F_{H_0}$ (see Definition 4.8) using some character of $H_0$.

The principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $X$ obtained by extending the structure group of $F_{H_0}$ using the projection $\phi$ in eqn. (4.10) is isomorphic to the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ in Definition 3.4.

**Proof.** Let $F_{\text{GL}(2,\mathbb{R})}$ be the principal $\text{GL}(2,\mathbb{R})$–bundle over $X$ obtained by extending the structure group of the principal $T_c$–bundle $F_{T_c}$ (see Definition 4.8) using the homomorphism $\tilde{\varphi}$ in eqn. (4.13). Let $V$ denote the vector bundle of rank two over $X$ associated to $F_{\text{GL}(2,\mathbb{R})}$ for the standard representation of $\text{GL}(2,\mathbb{R})$. Let

$$V_C := V \otimes_{\mathbb{R}} \mathbb{C}$$

be the vector bundle of rank two over the complexification $X_C$ obtained from $V$ by changing the base field to $\mathbb{C}$. From the construction of $F_{T_c}$ it follows immediately that the vector bundle $V_C$ is isomorphic to the direct sum $\zeta \oplus \sigma^* \zeta$ on $X_C$, where $\zeta$ is the ample generator of $\text{Pic}(X_C)$, and $\sigma$ is the anti–holomorphic involution of $X_C$ given by the action of the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$ on the complexification $X_C$.

Since $\text{deg}(\sigma^* \zeta) = \text{deg}(\zeta)$, and $\text{Pic}(X_C) = \mathbb{Z}$, we conclude that

$$\zeta \oplus \sigma^* \zeta = \zeta \oplus \zeta = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

on $X_C = \mathbb{P}^1$. Therefore,

$$V_C = \mathcal{O}_{X_C}(1) \oplus \mathcal{O}_{X_C}(1)$$

on $X_C = \mathbb{P}^1$.

We will show that the vector bundle $V$ on $X$ is indecomposable. To prove this, assume that $V$ decomposes as

$$V = \xi_1 \oplus \xi_2$$

into a direct sum of line bundles. We know that both $\xi_1$ and $\xi_2$ are tensor powers of the tangent bundle $TX$ (see Proposition 4.5). In particular, both degree$(\xi_1)$ and degree$(\xi_2)$ are even integers. For $i = 1, 2$, let

$$\xi_i := \xi_i \otimes_{\mathbb{R}} \mathbb{C}$$

be the complex algebraic line bundle over $X_C$ given by $\xi_i$. The decomposition in eqn. (4.17) gives a decomposition

$$V_C = (\xi_1) \oplus (\xi_2)_C$$

Comparing this decomposition with the one in eqn. (4.16), and using the Atiyah–Krull–Schmidt theorem on the uniqueness of the decomposition type of a vector bundle (see [At,
We conclude that
\[
\text{degree}(\xi_i) = \text{degree}((\xi_i)_C) = 1.
\]
This contradicts the earlier observation that the degree of $\xi_i$ is even. Therefore, we now conclude that the vector bundle $V$ is indecomposable.

Since $V$ is indecomposable, it can be shown that the principal $\text{GL}(2,\mathbb{R})$–bundle $F_{\text{GL}(2,\mathbb{R})}$ does not admit any reduction of structure group to $T_s = \mathbb{G}_m$. Indeed, any finite dimensional $\mathbb{G}_m$–module splits into a direct sum of one–dimensional modules, hence $V$ will decompose into a direct sum of line bundles if it is associated to a principal $T_s$–bundle. Therefore, the principal $H_0$–bundle $F_{H_0}$ does not admit any reduction of structure group to the subgroup $T_s \subset H_0$ in eqn. (4.10). This implies that the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{H_0}(\mathbb{Z}/2\mathbb{Z})$ over $X$ obtained by extending the structure group of $F_{H_0}$ using the projection in eqn. (4.10) is nontrivial. On the other hand, the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}/2\mathbb{Z}}$ (see Definition 3.4) is the unique nontrivial principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $X$ up to an isomorphism; this follows from Lemma 3.5. Therefore, the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{H_0}(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $F_{\mathbb{Z}/2\mathbb{Z}}$.

We will now show that all the line bundles over $X$ are associated to $F_{H_0}$. Since the group of all line bundles over $X$ is generated by the tangent bundle (see Proposition 4.5), it suffices to show that the tangent bundle $TX$ is associated to $F_{H_0}$ by some character of $H_0$.

Consider the character $\chi$ of $H_0$ defined by $g \mapsto \det \tilde{\varphi}(g) \in \mathbb{G}_m$, where $\tilde{\varphi}$ is the homomorphism defined in eqn. (4.13). This character coincides with the homomorphism $\nu$ in eqn. (4.14). It is easy to see that the line bundle over $X$ associated to $F_{H_0}$ for this character $\chi$ has degree two. Therefore, this associated line bundle is isomorphic to $TX$. This completes the proof of the proposition. 

**Remark 4.10.** The vector bundle $V_\mathbb{R}$ in [BN, p. 1213] is isomorphic to the vector bundle $V^*$ in the proof of Proposition 4.9. This follows immediately from the classification of vector bundles on $X$ given in [BN, p. 1208, Theorem 1.1].

**Proposition 4.11.** Let $E_{T_c}$ be a principal $T_c$–bundle over the nondegenerate anisotropic conic $X$. Then there is a homomorphism
\[
\beta : H_0 \longrightarrow T_c
\]
with the following property: The principal $T_c$–bundle over $X$ obtained by extending the structure group of the principal $H_0$–bundle $F_{H_0}$ (see Definition 4.8) using $\beta$ is isomorphic to $E_{T_c}$.

**Proof.** Fix a reduction of structure group $E_{H_0} \subset E_{T_c}$.
to the subgroup $H_0 \subset T_c$ (see Lemma 4.6). To prove the proposition it suffices to construct a homomorphism

$$h : H_0 \to H_0$$

such that the principal $H_0$--bundle $E_{H_0}$ is isomorphic to the one obtained by extending the structure group of the principal $H_0$--bundle $F_{H_0}$ using the homomorphism $h$.

Let $E_{\mathbb{Z}/2\mathbb{Z}}$ be the principal $\mathbb{Z}/2\mathbb{Z}$--bundle over $X$ obtained by extending the structure group of $E_{H_0}$ using the projection $\phi$ in eqn. (4.10). We first assume that $E_{\mathbb{Z}/2\mathbb{Z}}$ is a trivial principal $\mathbb{Z}/2\mathbb{Z}$--bundle. Then using the short exact sequence in eqn. (4.10) it follows that $E_{H_0}$ admits a reduction of structure group to the subgroup $T_s \subset H_0$. From the first part of Proposition 4.9 it follows immediately that there is homomorphism $h$ as in eqn. (4.18) which factors through the subgroup $T_s$ and satisfies the condition that $E_{H_0}$ is isomorphic to the principal $H_0$--bundle obtained by extending the structure group of $F_{H_0}$ using $h$.

Now assume that the above principal $\mathbb{Z}/2\mathbb{Z}$--bundle $E_{\mathbb{Z}/2\mathbb{Z}}$ is nontrivial. We note that $F_{\mathbb{Z}/2\mathbb{Z}}$ is the unique nontrivial principal $\mathbb{Z}/2\mathbb{Z}$--bundle on $X$ (see Lemma 3.5), and furthermore, $F_{\mathbb{Z}/2\mathbb{Z}}$ is the extension of structure group of $F_{H_0}$ by $\phi$ (see the second part of Proposition 4.9). Therefore, using the exact sequence in eqn. (4.10) we conclude that the two principal $H_0$--bundles $E_{H_0}$ and $F_{H_0}$ differ by a principal $T_s$--bundle. This means that there is a principal $T_s$--bundle $E'_{T_s}$ over $X$ such that the extension of structure group of the principal $T_s \times H_0$--bundle $E'_{T_s} \times_X F_{H_0}$ over $X$ using the multiplication homomorphism

$$T_s \times H_0 \to H_0$$

is isomorphic to $E_{H_0}$. Now the proof of the proposition is completed using the first part of Proposition 4.9. □

**Theorem 4.12.** Let $G$ be any connected reductive linear algebraic group defined over $\mathbb{R}$.

Let $E_G$ be a principal $G$--bundle over a nondegenerate anisotropic conic $X$. Then there is a homomorphism

$$\rho : H_0 \to G$$

such that $E_G$ is isomorphic to the principal $G$--bundle obtained by extending the structure group of the principal $H_0$--bundle $F_{H_0}$ using $\rho$ (see Definition 4.8 for $F_{H_0}$).

Let $F_{\mathbb{G}_m}$ denote the principal $\mathbb{G}_m$--bundle over $\mathbb{P}^1_\mathbb{R}$ corresponding to the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_\mathbb{R}}(1)$. Let $F_{\mathbb{G}_m} \times_X F_{\mathbb{Z}/2\mathbb{Z}}$ be the principal $\mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$--bundle over $X$ given by the fiber product (see Definition 3.4 for $F_{\mathbb{Z}/2\mathbb{Z}}$). Given any principal $G$--bundle $E_G$ over $\mathbb{P}^1_\mathbb{R}$, there is a homomorphism

$$\rho : \mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z}) \to G$$

such that $E_G$ is isomorphic to the principal $G$--bundle obtained by extending the structure group of the principal $\mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$--bundle $F_{\mathbb{G}_m} \times_X F_{\mathbb{Z}/2\mathbb{Z}}$ using $\rho$.

**Proof.** In view of Corollary 3.3, we may assume that $G$ is a torus.

In view of Lemma 4.1, it suffices to treat the three cases $T_s$, $T_a$ and $T_c$. 

Using the second part of Proposition 4.5 and the first part of Proposition 4.9 it follows that the theorem is valid for $G = T_s$. The second part of Proposition 4.9 implies that the theorem is valid for $G = T_a$.

For the anisotropic conic $X$, from Proposition 4.11 it follows that the statement in the theorem is valid for $G = T_c$. For $\mathbb{P}^1_\mathbb{R}$, the case of $T_c$ follows from Proposition 4.7 combined with the fact that the theorem is valid for $T_s$. This completes the proof of the theorem. □

References


School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Email address: indranil@math.tifr.res.in

Département de Mathématiques, Laboratoire CNRS UMR 6205, Université de Bretagne Occidentale, 6 avenue Victor Le Gorgeu, CS 93837, 29238 Brest cedex 3, France

Email address: johannes.huisman@univ-brest.fr