Scattering of linear Dirac fields
by a spherically symmetric Black-Hole

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Abstract - We study the linear Dirac system outside a spherical Black-Hole. In the case of massless fields, we prove the existence and asymptotic completeness of classical wave operators at the horizon of the Black-Hole and at infinity.

Résumé - On étudie le système linéaire de Dirac à l'extérieur d'un Trou Noir sphérique. Dans le cas des champs sans masse, on montre l'existence et la complétude asymptotique des opérateurs d'onde classiques à l'horizon du Trou Noir et à l'infini.

1 Introduction

We develop a time-dependent scattering theory for the linear Dirac system on Schwarzschild-type metrics. The first time-dependent scattering results on the Schwarzschild metric were obtained by J. Dimock [8]. Using the short range at infinity of the interaction between gravity and a massless scalar field, he proved the existence and asymptotic completeness of classical wave-operators for the wave equation. The case of the Maxwell system in which the interaction is pseudo long-range has been worked out by A. Bachelot [2], and for the Regge-Wheeler equation, a complete scattering theory has been developed by A. Bachelot and A. Motet-Bachelot [3]. Our purpose in this work is to study the classical wave operators and their asymptotic completeness for the linear massless Dirac system on a general "Schwarzschild-type" metric which covers all the usual cases of spherical black-holes. The main tools are Cook’s method for the existence and the results obtained in [3] for the asymptotic completeness.

Let us consider the manifold $\mathbb{R}_t \times ]0, +\infty[ r \times S^2_{\theta, \phi}$ endowed with the pseudo-riemannian metric

$$g_{\mu \nu} dx^\mu dx^\nu = F(r)e^{2\kappa(r)} dt^2 - [F(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2]$$

(1)

where $F, \kappa \in C^\infty(]0, +\infty[)$. We assume the existence of three values $r_\nu$ of $r$, $0 \leq r_- < r_0 < r_+ \leq +\infty$, which are the only possible zeros of $F$, such that

$$F(r_\nu) = 0 \ , \ F'(r_\nu) = 2\kappa_\nu \ , \ \kappa_\nu \neq 0 \ , \ \text{if } 0 < r_\nu < +\infty,$$

$$F(r) > 0 \text{ for } r \in ]r_0, r_+[, \ F(r) < 0 \text{ for } r \in ]r_-, r_0[.$$

When they are finite and non zero, $r_-, r_0$ and $r_+$ are the radii of the spheres called: horizon of the black-hole ($r_0$), Cauchy horizon ($r_-$) and cosmological horizon ($r_+$). $\kappa_\nu$ is the surface gravity at the horizon $\{r = r_\nu\}$. If $r_+$ is infinite, we assume moreover that

$$F(r) = 1 - \frac{\Delta}{r} + O\left(r^{-2}\right) \ , \ r_1 > 0 \ , \ \delta(r) = \delta(+\infty) + o(r^{-1}) \ , \ r \to +\infty,$$

$$F'(r) , \ \delta'(r) = O(r^{-2}) \ , \ r \to +\infty.$$
All these properties are satisfied by usual spherical black-holes (see [13]).

**Notations:** Let \((M, g)\) be a Riemannian manifold, \(C_0^\infty(M)\) denotes the set of \(C^\infty\)functions with compact support in \(M\), \(H^k(M, g), k \in \mathbb{N}\) is the Sobolev space, completion of \(C_0^\infty(M)\) for the norm
\[
\|f\|_{H^k(M)}^2 = \sum_{j=0}^k \int_M \langle \nabla^j f, \nabla^j f \rangle \, d\mu,
\]
where \(\nabla^j\), \(d\mu\) and \(<,>\) are respectively the covariant derivatives, the measure of volume and the hermitian product associated with the metric \(g\). We write \(L^2(M, g) = H^0(M, g)\).

If \(E\) is a distribution space on \(M\), \(E_{\text{comp}}\) represents the subspace of elements of \(E\) with compact support in \(M\).

The 2-dimensional euclidian sphere \(S^2\) is endowed with its usual metric
\[
d\omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi.
\]

## 2 The covariant generalization of the linear Dirac system on Schwarzschild-type metrics

The covariant generalization of the Dirac system on the metric \(g\) has the form
\[
(i\gamma^\mu \nabla_\mu - m) \Phi = 0, \quad m \geq 0 \tag{2}
\]
for a particle with mass \(m\), where \(\Phi\) is a Dirac 4-spinor, the \(\gamma^\mu\) are the contravariant Dirac matrices on curved space-time and \(\nabla_\mu\) is the covariant derivation of spinor fields. We make the following choices of flat space-time Dirac matrices
\[
\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma_\alpha = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix}, \quad \alpha = 1, 2, 3 \tag{3}
\]
where
\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}
\]
are the Pauli matrices, and of local Lorentz frame
\[
e_{\tilde{\alpha}}^\mu = \begin{cases} \frac{|g^\mu\nu|}{2} & \text{if} \quad \tilde{\alpha} = \nu, \\ 0 & \text{if} \quad \tilde{\alpha} \neq \nu. \end{cases} \tag{5}
\]
We recall that flat space time Dirac matrices are a set of 4x4 matrices \(\{\gamma_\tilde{\alpha}\}_{0 \leq \tilde{\alpha} \leq 3}\) such that
\[
\{\gamma_\tilde{\alpha}, \gamma_\tilde{\beta}\} = \gamma_\tilde{\alpha} \gamma_\tilde{\beta} + \gamma_\tilde{\beta} \gamma_\tilde{\alpha} = 2\eta_{\tilde{\alpha}\tilde{\beta}} \mathbb{I} \quad (\tilde{\alpha}, \tilde{\beta} = 0, 1, 2, 3) \tag{6}
\]
where
\[
\eta_{\tilde{\alpha}\tilde{\beta}} = \text{diag}(1, -1, -1, -1) \tag{7}
\]
is the Minkowski metric. The indices with a tilde refer to flat space-time and can be raised or lowered using \(\eta_{\tilde{\alpha}\tilde{\beta}}\), whereas the indices without tilde refer to curved space-time and are raised or lowered using the metric \(g\).

With these definitions, the \(\gamma^\mu\) and \(\nabla_\mu\) are then defined by (see for example [5], [7])
\[
\gamma^\mu = \gamma_\tilde{\alpha} e^{\tilde{\alpha}_\mu} \tag{8}
\]
and
\[
\nabla_\mu = \partial_\mu + \frac{1}{2} G_{[\tilde{\alpha}\tilde{\beta}]\mu}^{\tilde{\alpha}\tilde{\beta}} \tag{9}
\]
where
\[
G_{[\tilde{\alpha}\tilde{\beta}]} = \frac{1}{4} \left[ \gamma_\tilde{\alpha}, \gamma_\tilde{\beta} \right] = \frac{1}{4} \left( \gamma_\tilde{\alpha} \gamma_\tilde{\beta} - \gamma_\tilde{\beta} \gamma_\tilde{\alpha} \right) \tag{10}
\]
are the generators of the spinor representation of the proper Lorentz group and

\[ \omega^{\delta \beta}_{\mu} = \frac{1}{2} e^{\bar{\delta} \nu} (e^{\delta \nu}_{\mu} - e^{\beta \nu}_{\mu}) - \frac{1}{2} e^{\bar{\beta} \nu} (e^{\delta \nu}_{\mu} - e^{\epsilon \nu}_{\mu}) + \frac{1}{2} e^{\bar{\epsilon} \nu} e^{\bar{\gamma} \sigma} (e^{\epsilon \nu \sigma} - e^{\delta \nu \epsilon}) e_{\gamma \mu} = -\omega^{\delta \beta}_{\mu} \]  

(11)

are the coefficients of the spin connection, \( \mu \) standing for the derivation with respect to the \( \mu \)-th variable. We compute the a priori non zero components:

\[ \omega^{\bar{e} \bar{t}} = \frac{1}{2} e^{\bar{t}} [\partial_t (e^{\bar{t}}) - \partial_t (e^{\bar{t}})] - \frac{1}{2} e^{\bar{t}} e^r [\partial_t (e^{\bar{r}}) - \partial_t (e^{\bar{r}})] + \frac{1}{2} e^{\bar{t}} e^\varphi [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] e_{tt} = \frac{1}{2} \left( 2 F^t + F^t \right)^2 e^t, \]

\[ \omega^{\bar{e} \bar{r}} = \frac{1}{2} e^{\bar{r}} [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] - \frac{1}{2} e^{\bar{r}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{r}} e^\varphi [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] e_{rr} = 0, \]

\[ \omega^{\bar{e} \bar{\theta}} = \frac{1}{2} e^{\bar{\theta}} [\partial_\theta (e^{\bar{\theta}}) - \partial_\theta (e^{\bar{\theta}})] - \frac{1}{2} e^{\bar{\theta}} e^r [\partial_\theta (e^{\bar{r}}) - \partial_\theta (e^{\bar{r}})] + \frac{1}{2} e^{\bar{\theta}} e^\varphi [\partial_\varphi (e^{\bar{\theta}}) - \partial_\varphi (e^{\bar{\theta}})] e_{\theta \theta} = 0, \]

\[ \omega^{\bar{e} \bar{\varphi}} = \frac{1}{2} e^{\bar{\varphi}} [\partial_\varphi (e^{\bar{\varphi}}) - \partial_\varphi (e^{\bar{\varphi}})] - \frac{1}{2} e^{\bar{\varphi}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{\varphi}} e^\varphi [\partial_\varphi (e^{\bar{\varphi}}) - \partial_\varphi (e^{\bar{\varphi}})] e_{\varphi \varphi} = 0, \]

\[ \omega^{\bar{r} \bar{r}} = \frac{1}{2} e^{\bar{r}} [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] - \frac{1}{2} e^{\bar{r}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{r}} e^\varphi [\partial_\varphi (e^{\bar{r}}) - \partial_\varphi (e^{\bar{r}})] e_{rr} = 0, \]

\[ \omega^{\bar{r} \bar{\theta}} = \frac{1}{2} e^{\bar{\theta}} [\partial_\theta (e^{\bar{r}}) - \partial_\theta (e^{\bar{r}})] - \frac{1}{2} e^{\bar{\theta}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{\theta}} e^\varphi [\partial_\varphi (e^{\bar{r}}) - \partial_\varphi (e^{\bar{r}})] e_{r \theta} = 0, \]

\[ \omega^{\bar{r} \bar{\varphi}} = \frac{1}{2} e^{\bar{\varphi}} [\partial_\varphi (e^{\bar{r}}) - \partial_\varphi (e^{\bar{r}})] - \frac{1}{2} e^{\bar{\varphi}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{\varphi}} e^\varphi [\partial_\varphi (e^{\bar{r}}) - \partial_\varphi (e^{\bar{r}})] e_{r \varphi} = 0, \]

\[ \omega^{\bar{\theta} \bar{\theta}} = \frac{1}{2} e^{\bar{\theta}} [\partial_\theta (e^{\bar{\theta}}) - \partial_\theta (e^{\bar{\theta}})] - \frac{1}{2} e^{\bar{\theta}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{\theta}} e^\varphi [\partial_\varphi (e^{\bar{\theta}}) - \partial_\varphi (e^{\bar{\theta}})] e_{\theta \theta} = 0, \]

\[ \omega^{\bar{\theta} \bar{\varphi}} = \frac{1}{2} e^{\bar{\varphi}} [\partial_\varphi (e^{\bar{\theta}}) - \partial_\varphi (e^{\bar{\theta}})] - \frac{1}{2} e^{\bar{\varphi}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{\varphi}} e^\varphi [\partial_\varphi (e^{\bar{\theta}}) - \partial_\varphi (e^{\bar{\theta}})] e_{\theta \varphi} = 0, \]

\[ \omega^{\bar{\varphi} \bar{\varphi}} = \frac{1}{2} e^{\bar{\varphi}} [\partial_\varphi (e^{\bar{\varphi}}) - \partial_\varphi (e^{\bar{\varphi}})] - \frac{1}{2} e^{\bar{\varphi}} e^r [\partial_r (e^{\bar{r}}) - \partial_r (e^{\bar{r}})] + \frac{1}{2} e^{\bar{\varphi}} e^\varphi [\partial_\varphi (e^{\bar{\varphi}}) - \partial_\varphi (e^{\bar{\varphi}})] e_{\varphi \varphi} = 0, \]

and we obtain the following expression for the linear massive Dirac equation outside a spherical black-hole:

\[ \left\{ \gamma^\mu \partial_\mu + \mu e^{\beta} \gamma^1 \left( \partial_\tau + \frac{1}{r} + \frac{F^t}{2} + \frac{F^t e^{\delta}}{2} \right) + \frac{F^t e^{\delta}}{r \sin \theta} \right\} \partial_{\beta} \Phi = 0. \]

(12)

We introduce the frame with respect to which we shall express the equation, \( R' = \left( \frac{1}{r \sin \theta}, \frac{1}{\cos \theta}, \frac{1}{\sin \theta} \frac{1}{\tan \theta} \partial_\varphi \right) \), image of \( R = (F^t \partial_t, \frac{1}{\sin \theta} \partial_\theta, \frac{1}{\cos \theta} \partial_\varphi) \) by the spatial rotation \( f \) with Euler angles (see for example [15]) \( (\varphi, \theta, \psi) = (0, \pi/2, \pi) \), and the Regge-Wheeler variable \( r_* \) defined by

\[ \frac{dr}{dr_*} = \mu e^{\delta}, \quad \tau \in [r_0, r_*]. \]

(13)

The spinor

\[ \Psi = T_{(f^{-1})} r^1 e^{\delta/2} \Phi, \]

(14)
Thus, for any half-integer \( m \), we know from [12], that

\[
\int_{\mathbb{R}} T_{-1} \frac{e^{\gamma_2 \varphi}}{r} \gamma_2 \varphi \left( \partial_\varphi + \frac{1}{2} \cotg \theta \right) + \frac{e^{\gamma_2 \varphi}}{r \sin \theta} \gamma_2 \varphi \partial_\varphi + i \gamma_2 \partial_\varphi + F^{1/2} e^{\gamma_2 \varphi} m \]

(15)
on the domain \( \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega \) representing the exterior of the black-hole in the variables \((t,r,\omega)\).

We recall (see [7]) that, given a spatial rotation \( f \) of angle \( \theta \) around a unit vector \( n = (n_1,n_2,n_3) \), its associated spin transformation \( T_f \) is

\[
T_f = \text{Exp} \left\{ \left[ n_1 G_{[2,3]} + n_2 G_{[3,1]} + n_3 G_{[1,3]} \right] \theta \right\}
\]

where \( \text{Exp} \) is the exponential mapping.

### 3 Global Cauchy problem

We introduce the Hilbert space

\[ \mathcal{H} = \{ L^2 (\mathbb{R}_r \times S^2_\omega; dr^2 + d\omega^2) \}^4. \]

**Theorem 3.1.** Given \( \Psi_0 \in \mathcal{H} \), equation (15) has a unique solution \( \Psi \) such that

\[ \Psi \in \mathcal{C} (\mathbb{R}_t; \mathcal{H}) \text{ , } \Psi |_{t=0} = \Psi_0. \]

Moreover, for any \( t \in \mathbb{R} \)

\[ \| \Psi(t) \|_{\mathcal{H}} = \| \Psi_0 \|_{\mathcal{H}}. \]

**Proof:** We show that the operator

\[ \hat{H} = H + \gamma_2^2 e^{\gamma_2 \varphi} \]

is self-adjoint with dense domain on \( \mathcal{H} \). We decompose \( \mathcal{H} \) using generalized spherical functions of weights \( 1/2 \) and \(-1/2\). Let

\[ \mathcal{I} = \{(l,m,n) \ ; \ 2l,2m,2n \in \mathbb{Z} \ ; \ l - |m|, l - |n| \in \mathbb{N} \} \]

and for any half-integer \( m \)

\[ \mathcal{I}_m = \{(l,n) \ ; \ (l,m,n) \in \mathcal{I} \}. \]

For \((l,m,n) \in \mathcal{I} \), we define the function \( T^l_{mn}(\varphi_1,\varphi_2) \), \( \varphi_1,\varphi_2 \in [0,2\pi[, \theta \in [0,\pi] \), by

\[ T^l_{mn}(\varphi_1,\varphi_2) = e^{-im\varphi_2} u'^{l}_{mn}(\theta) e^{-in\varphi_1} \]

(23)

where \( u'^{l}_{mn} \) satisfies the following ordinary differential equations

\[
\frac{d^2 u'^{l}_{mn}}{d\theta^2} + \cotg \theta \frac{du'^{l}_{mn}}{d\theta} + \left[ \frac{l(l+1)}{} - \frac{n^2 - 2mn \cos \theta + m^2}{\sin^2 \theta} \right] u'^{l}_{mn} = 0,
\]

(24)

\[
\frac{du'^{l}_{mn}}{d\theta} = \frac{n - m \cos \theta}{\sin \theta} u'^{l}_{mn} = -i \left[ (l+m)(l-m+1) \right]^{1/2} u'^{l}_{m-1,n},
\]

(25)

\[
\frac{du'^{l}_{mn}}{d\theta} = \frac{n - m \cos \theta}{\sin \theta} u'^{l}_{mn} = -i \left[ (l+m+1)(l-m) \right]^{1/2} u'^{l}_{m+1,n},
\]

(26)

and the normalization condition

\[
\int_0^\pi |u'^{l}_{mn}(\theta)|^2 \sin \theta d\theta = \frac{1}{4\pi^2}.
\]

(27)

We know from [12], that \( \{ T^l_{mn} \}_{(l,m,n) \in \mathcal{I}_m} \) is a Hilbert basis of

\[ L^2 \left( [0,2\pi[ \times [0,\pi] \times [0,2\pi[; \sin^2 \theta d\varphi_1^2 + d\theta^2 + d\varphi_2^2 \right). \]

(28)

Thus, for any half-integer \( m \),

\[ \{ T^l_{mn}(\varphi,\theta,0) = e^{-im\varphi} u'^{l}_{mn}(\theta) \}_{(l,n) \in \mathcal{I}_m} \]
is a Hilbert basis of $L^2(S^n; d\omega^2)$. In particular,

$$\mathcal{H} = \bigoplus_{(l,n)\in \mathcal{I}_\frac{1}{2}} \mathcal{H}_{ln}$$

(29)

where

$$\mathcal{H}_{ln} = \left\{ t \left( f_1 T_{\frac{1}{2},n}^l, f_2 T_{\frac{1}{2},n}^l, f_3 T_{-\frac{1}{2},n}^l, f_4 T_{-\frac{1}{2},n}^l \right) ; f_i \in L^2(\mathbb{R}^r; dr_r^2), \ i = 1, 2, 3, 4 \right\}, \quad (26)$$

or equivalently,

$$\mathcal{H}_{ln} = \left[ L^2(\mathbb{R}^r; dr_r^2) \right]^4 \otimes F_{ln} \quad \text{or} \quad F_{ln} = t \left( T_{\frac{1}{2},n}^l, T_{\frac{1}{2},n}^l, T_{-\frac{1}{2},n}^l, T_{-\frac{1}{2},n}^l \right) \quad (30)$$

(31)

where the $T_{\pm\frac{1}{2},n}^l$ are seen as functions of only $\varphi, \theta$. Let

$$\Psi = t (f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln}.$$ 

Denoting $\alpha = F^{1/2}e^\delta$, the four components of $\hat{H}\Psi$ are

$$i\partial_r f_3 T_{\frac{1}{2},n}^l - \frac{\alpha}{r} f_4 \left( \partial_\theta + \frac{1}{2} \cot g \theta \right) T_{\frac{1}{2},n}^l + i \frac{\alpha}{r \sin \theta} f_4 \partial_\varphi T_{\frac{1}{2},n}^l,$$

$$-i\partial_r f_4 T_{\frac{1}{2},n}^l + \frac{\alpha}{r} f_3 \left( \partial_\theta + \frac{1}{2} \cot g \theta \right) T_{\frac{1}{2},n}^l + i \frac{\alpha}{r \sin \theta} f_3 \partial_\varphi T_{\frac{1}{2},n}^l,$$

$$i\partial_r f_1 T_{-\frac{1}{2},n}^l - \frac{\alpha}{r} f_2 \left( \partial_\theta + \frac{1}{2} \cot g \theta \right) T_{-\frac{1}{2},n}^l + i \frac{\alpha}{r \sin \theta} f_2 \partial_\varphi T_{-\frac{1}{2},n}^l,$$

$$-i\partial_r f_2 T_{-\frac{1}{2},n}^l + \frac{\alpha}{r} f_1 \left( \partial_\theta + \frac{1}{2} \cot g \theta \right) T_{-\frac{1}{2},n}^l + i \frac{\alpha}{r \sin \theta} f_1 \partial_\varphi T_{-\frac{1}{2},n}^l.$$ 

Relations (25) and (26) yield

$$\left( \partial_\theta + \frac{1}{2} \cot g \theta \right) T_{\frac{1}{2},n}^l = \frac{n}{\sin \theta} T_{\frac{1}{2},n}^l - i \left( l + \frac{1}{2} \right) T_{-\frac{1}{2},n}^l, \quad (32)$$

$$\left( \partial_\theta + \frac{1}{2} \cot g \theta \right) T_{-\frac{1}{2},n}^l = -\frac{n}{\sin \theta} T_{-\frac{1}{2},n}^l - i \left( l + \frac{1}{2} \right) T_{\frac{1}{2},n}^l, \quad (33)$$

and we also have

$$\partial_\varphi T_{\pm\frac{1}{2},n}(\varphi, \theta, 0) = -inT_{\pm\frac{1}{2},n}(\varphi, \theta, 0). \quad (34)$$

Thus, the four components of $\hat{H}\Psi$ are

$$(i\partial_r f_3 + \frac{n}{r} (l + \frac{1}{2}) f_4) T_{-\frac{1}{2},n}^l,$$

$$(-i\partial_r f_4 - \frac{n}{r} (l + \frac{1}{2}) f_3) T_{-\frac{1}{2},n}^l,$$

$$(i\partial_r f_1 + \frac{n}{r} (l + \frac{1}{2}) f_2) T_{\frac{1}{2},n}^l,$$

$$(-i\partial_r f_2 - \frac{n}{r} (l + \frac{1}{2}) f_1) T_{\frac{1}{2},n}^l.$$ 

We see that on $\mathcal{H}_{ln}$, $\hat{H}$ has the form

$$\hat{H} |_{\mathcal{H}_{ln}} = \left( i\partial_r L + \frac{\alpha}{r} \left( l + \frac{1}{2} \right) M \right)_{r, \varphi} \otimes \mathbb{I}_\theta \varphi$$

(35)

where the matrices $L$ et $M$, defined by

$$L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \quad (36)$$
are hermitian and \( L \) is invertible. Since the function \( \alpha r^{-1} \) belongs to \( L^\infty(\mathbb{R}_r^+) \), \( \tilde{H} \mid_{\mathcal{H}_{in}} \) is self-adjoint with domain
\[
D_{in} = [D(i\partial_r^*)]^4 \otimes F_{in} \simeq [H^4(\mathbb{R}_r^+; dr^2)]^4 \otimes F_{in}
\]
dense in \( \mathcal{H}_{in} \). On \( D_{in} \), we choose the following norm
\[
\|\Psi\|_{D_{in}}^2 = \|\Psi\|_{(L^2(\mathbb{R}))^4}^2 + \left\| \left( i\partial_r \right. \right. L + \frac{\alpha}{r} \left( l + \frac{1}{2} \right) M \left. \right) \Psi \|_{(L^2(\mathbb{R}))^4}^2
\]
and we introduce the dense subspace of \( \mathcal{H} \)
\[
D(H) = \left\{ \Psi = \sum_{(l,n)} \Psi_{ln} ; \Psi_{ln} \in D_{in} \right. , \sum_{(l,n)} \|\Psi_{ln}\|_{D_{in}}^2 < +\infty \right\}.
\]
\( \tilde{H} \) is self-adjoint on \( \mathcal{H} \) with domain \( D(H) \), \( \gamma^0 \alpha m \) is self-adjoint and bounded on \( \mathcal{H} \), therefore, \( H \) is self-adjoint on \( \mathcal{H} \) with dense domain \( D(H) \). Theorem 3.1 follows from Stone’s theorem.

\[Q.E.D.\]

4 Wave operators at the horizon

When \( r \to r_0 \), the operator \( H \) has the formal limit
\[
H_0 = i\gamma^5 \rho \gamma^3 \partial_r
\]
which is a self-adjoint operator on \( \mathcal{H} \) with dense domain
\[
D(H_0) = \{ H^4 \left( \left( \mathbb{R}_r^+ ; dr^2 \right) ; L^2 \left( S^2_r ; dw^2 \right) \right) \}^4.
\]
The spectrum of \( H_0 \) is purely absolutely continuous. We define the subspaces of incoming and outgoing waves associated with \( H_0 \):
\[
\mathcal{H}_0^\pm = \{ \Psi = \iota (u^1, u^2, u^3, u^4) \mid u^3 = \mp u^1, \ u^4 = \pm u^2 \}. \tag{42}
\]
\( \mathcal{H}_0^\pm \) as well as the \( \mathcal{H}_{in} \) remain stable under \( H_0 \) and we have
\[
\mathcal{H} = \mathcal{H}_0^+ \oplus \mathcal{H}_0^- , \quad \forall \Psi_0 \in \mathcal{H}_0^\pm , \quad (e^{iH_0 t} \Psi_0)(r_+, \omega) = \Psi_0(r_\pm \mp t, \omega).
\]
Since we want to compare \( H \) with \( H_0 \) in the neighbourhood of the horizon, we introduce the cut-off function
\[
\chi_o \in C^\infty(\mathbb{R}_r^+) , \quad 0 \leq \chi_o \leq 1, \quad \exists a, b \in \mathbb{R} , \quad a < b \ such \ that \forall r_* < a, \chi_o(r_*) = 1 \quad \forall r_* > b, \chi_o(r_*) = 0
\]
for \( \rho = \alpha \), \( \chi_o(r_\rho) = 1 \); for \( \rho > b \), \( \chi_o(r_\rho) = 0 \)

\[Q.E.D.\]

Theorem 4.1. The operator \( W_0^+ \) (resp. \( W_0^- \)) is well-defined from \( \mathcal{H}_0^+ \) (resp. \( \mathcal{H}_0^- \)) to \( \mathcal{H} \), is independent of the choice of \( \chi_o \), satisfying (44), moreover
\[
\forall \Psi_0 \in \mathcal{H}_0^\pm , \quad \| W_0^\pm \Psi_0 \|_{\mathcal{H}} = \| \Psi_0 \|_{\mathcal{H}}.
\]

\[\text{Theorem 4.1.}\]
Remark 4.1. In the case where the identifying operator $J$ associated identifying operators $t$ tends to infinity, which gives (47). If now we consider two different cut-off functions $\chi$ and $\chi'$, and the associated identifying operators $J_0$ and $J'_0$, the difference $\chi - \chi'$ is compactly supported, thus

$$\|e^{-iHt}J_0 e^{iHt}\Psi_0 - e^{-iHt}J'_0 e^{iHt}\Psi_0\|_{\mathcal{H}} \to 0, \quad t \to \pm \infty.$$  

Q.E.D.

Remark 4.1. In the case where $r_+$ is finite, we construct in the same way classical wave operators at the cosmological horizon

$$W_1^+ \Psi_0 = \lim_{t \to \pm \infty} e^{-iHt}J_1 e^{iHt}\Psi_0 \quad \text{in } \mathcal{H}$$

where the identifying operator $J_1$ is defined by

$$J_1 : \mathcal{H} \Psi \mapsto \chi_1 \Psi,$$

$\chi_1$ being a cut-off function

$$\chi_1 \in C^\infty(\mathbb{R}_{r_+}), \quad 0 \leq \chi_1 \leq 1,$$

$$\exists a, b \in \mathbb{R}, \quad a < b \quad \text{such that}$$

$$\text{for } r_* < a \quad \chi_1(r_*) = 0; \quad \text{for } r_* > b \quad \chi_1(r_*) = 1.$$

$W_1^+$ (resp. $W_1^-$) is an isometry from $\mathcal{H}_0^+$ (resp. $\mathcal{H}_0^-$) to $\mathcal{H}$ and is independent of the choice of $\chi_1$ satisfying (53).
5 Wave operators at infinity (massless case)

In all this paragraph, we shall assume that \( r_+ = +\infty \); the metric (1) is then asymptotically flat in the neighbourhood of infinity and we choose to compare \( H \) to an operator \( H_\infty \) which is equivalent to the hamiltonian operator for the Dirac equation on the Minkowski space-time. We also make the hypothesis that \( m = 0 \) in order to avoid long range perturbations at infinity. Let us consider on the Minkowski metric
\[
ds_M^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad ; \quad x, y, z \in \mathbb{R}
\]
the massless Dirac system
\[
\left\{ \gamma^0 \partial_t + \gamma^1 \partial_x + \gamma^2 \partial_y + \gamma^3 \partial_z \right\} \Phi = 0.
\]
The associated hamiltonian operator, defined by
\[
H_M = i \gamma^0 \left\{ \gamma^1 \partial_x + \gamma^2 \partial_y + \gamma^3 \partial_z \right\},
\]
is self-adjoint with dense domain on \( [L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4 \) and if \( \Phi \in C(\mathbb{R}_t; [L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4) \) is a solution of (55), its energy in a compact domain goes to zero when \( t \) goes to \( \pm \infty \). In addition, for any \( \Phi_0 \in [L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4 \) with a compact support contained in
\[
B(0, R) = \left\{ (x, y, z); 0 \leq \rho < R \ , \ \rho = (x^2 + y^2 + z^2)^{1/2} \right\},
\]
the solution \( \Phi \) of (55) associated with the initial data \( \Phi_0 \) satisfies
\[
\Phi(t, x, y, z) = 0 \quad \text{for} \quad 0 \leq \rho \leq |t| - R.
\]

At the point of spherical coordinates \( (\rho, \theta, \varphi) \), we apply the spatial rotation \( f \) with Euler angles \( (\pi/2, \theta, \pi-\varphi) \). The local frame \( (\partial_x, \partial_y, \partial_z) \) is thus transformed by \( f^{-1} \) into
\[
(\partial_{x'}, \partial_{x''}, \partial_{x'''} ) = \left( \frac{1}{\rho \sin \theta} \partial_\varphi, -\frac{1}{\rho} \partial_\theta, \partial_\rho \right).
\]
The spinor
\[
\Psi = \rho T_f \Phi,
\]
where \( T_f \) is the spin transformation associated with \( f \) defined in (16), satisfies
\[
\partial_t \Psi = i H_\infty \Psi \quad , \quad H_\infty = i \left[ \gamma^0 \gamma^2 \partial_\rho + \frac{1}{\rho} \gamma^0 \frac{1}{2} \cot \theta \partial_\theta + \frac{1}{\rho \sin \theta} \gamma^0 \gamma^1 \partial_\varphi \right].
\]
The operator \( H_\infty \) on
\[
\mathcal{H}_\infty = \left\{ L^2[0, +\infty; S^2_2 \times S^2_2 ; \ d\rho^2 + d\omega^2] \right\}^4
\]
is unitarily equivalent to \( H_M \) on
\[
\left\{ L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z ; \ d\rho^2 + d\omega^2 + dz^2) \right\}^4.
\]
Therefore, \( H_\infty \) is self-adjoint with dense domain on \( \mathcal{H}_\infty \) and if \( \Phi \in C(\mathbb{R}_t; \mathcal{H}_\infty) \) satisfies (61), then its energy in a compact domain goes to zero when \( t \) goes to \( \pm \infty \). Moreover, for
\[
\Psi_0 \in \mathcal{H}_\infty \ ; \ \text{Supp}(\Psi_0) \subset B(0, R)
\]
\[
\Psi(t) = e^{i H_\infty t} \Psi_0 \text{ satisfies}
\]
\[
\Psi(t, \rho, \theta, \varphi) = 0 \quad \text{for} \quad 0 \leq \rho \leq |t| - R.
\]

In order to avoid artificial long-range interactions, we choose
\[
\rho = r_+ \geq 0
\]
and we introduce the cut-off function
\[ \chi_\infty \in C^\infty ([0, +\infty[, r_*), \quad 0 \leq \chi_\infty \leq 1, \]

for \( 0 \leq r_* \leq a \quad \chi_\infty(r_*) = 0 \), \( for \ r_* \geq b \quad \chi_\infty(r_*) = 1 \)

with the identifying operator
\[ J_\infty : \mathcal{H}_\infty \rightarrow \mathcal{H} : \quad \forall \Psi \in \mathcal{H}_\infty \]

\[ (J_\psi)_{|\{r_* \leq 0\}} = \chi_\infty \Psi, \]

\[ (J_\psi)_{|\{r_* \geq 0\}} = 0. \]

(65)

Theorem 5.1. The operators \( W^\pm_\infty \) are well-defined from \( \mathcal{H}_\infty \) to \( \mathcal{H} \), are independent of the choice of \( \chi_\infty \) and

\[ \forall \Psi_0 \in \mathcal{H}_\infty, \quad \| W^\pm_\infty \Psi_0 \|_{\mathcal{H}} = \| \Psi_0 \|_{\mathcal{H}_\infty}. \]

(66)

Proof: For \( (l,n) \in \mathcal{I}_{1/2} \), we introduce the subspaces of \( \mathcal{H}_\infty \)

\[ D^\infty_{ln} = \{ \Psi = (f_1, f_2, f_3, f_4) \otimes F_{in} \in \mathcal{H}_\infty; \ 1 \leq i \leq 4 \ f_i \in C^\infty_0 (\mathbb{R}_{r_*}^+) \} \]

the direct sum of which is dense in \( \mathcal{H}_\infty \). For \( \Psi_0 \in D^\infty_{ln}, \)

\[ H_\infty |D^\infty_{ln} = \left( i\partial_{r_*} L + \frac{1}{r_*} \left( l + \frac{1}{2} \right) M \right)_{r_*} \otimes \mathbb{I} \]

where the matrices \( L \) and \( M \) are defined by (36), and

\[ J_\infty \Psi_0 \in \mathcal{H}_{ln}. \]

(70)

\( J_\infty \) being a bounded operator, it suffices to prove that for

\[ \Psi_0 \in D^\infty_{ln}; \quad \text{Supp}(\Psi_0) \subset B(0, R), \]

we have

\[ \| (H J_\infty - J_\infty H_\infty) e^{iH_\infty t} \Psi_0 \|_{\mathcal{H}} \in L^1(\mathbb{R}_t). \]

(73)

(63) yields

\[ e^{iH_\infty t} \Psi_0 = 0 \quad \text{in} \quad \{(l, r_* , \theta, \psi) ; \ 0 \leq r_* \leq |t| - R \}. \]

(74)

Thus, for \( |t| \) large enough

\[ \| (H J_\infty - J_\infty H_\infty) e^{iH_\infty t} \Psi_0 \|_{\mathcal{H}} = \| \left( \frac{\alpha}{r} - \frac{1}{r_*} \right) \left( l + \frac{1}{2} \right) M e^{iH_\infty t} \Psi_0 \|_{\mathcal{H}} \]

\[ \leq \left( l + \frac{1}{2} \right) \| \Psi_0 \|_{\mathcal{H}_\infty} \left\| \frac{\alpha}{r} - \frac{1}{r_*} \right\|_{L^\infty(|t|+R, +\infty; [r_*])}. \]

We study the asymptotic behavior of

\[ \frac{\alpha}{r} - \frac{1}{r_*} = \frac{1}{r_*} \left( F^{1/2} e^{\delta r_0} \frac{r}{r_*} - 1 \right) \]

when \( r_* \) goes to \( +\infty \). The Regge-Wheeler variable \( r_* \) is defined with respect to \( r \) by

\[ r_* = \frac{1}{2\kappa_0} \left\{ \log |r - r_0| - \int_{r_0}^r \left[ \frac{1}{r - r_0} - \frac{2\kappa_0}{F e^\delta} \right] d \right\} \]

(75)
where \(2\kappa_0 = F'(r_0)\). For \(r\) larger than \(r_0 + 1\), we have

\[
r_* = C + \int_{r_0 + 1}^{r} F^{-1} e^{-\delta} dr
\]

(76)

where

\[
2\kappa_0 C = - \int_{r_0}^{r_0 + 1} \frac{1}{r - r_0} - \frac{2\kappa_0}{Fe^r} dr.
\]

(77)

\(F\) and \(\delta\) satisfy

\[
\delta(r) = o(r^{-1}) ; \quad F(r) = 1 - \frac{r_1}{r} + O(r^{-2}) \quad r_1 > 0 ; \quad r \to +\infty
\]

and therefore

\[
F^{-1}(r)e^{-\delta(r)} = 1 + \frac{r_1}{r} + o(r^{-1}),
\]

\[
r_* = r + r_1 \text{Log}(r) + o(\text{Log}(r)),
\]

\[
F^{1/2}(r)e^{\delta(r)} = 1 - \frac{r_1}{2r} + o(r^{-1})
\]

which implies

\[
F^{1/2}(r)e^{\delta(r)} \frac{r_*}{r} - 1 = r_1 \frac{\text{Log}(r)}{r} + o \left( \frac{\text{Log}(r)}{r} \right) = O(r^{-1/2}) = O(r_*^{-1/2}).
\]

The operators \(W_{0, \infty}^{\pm}\) are thus well-defined. The fact that they are isometries and do not depend on the choice of the cut-off function can be verified using exactly the same remarks as in the case of the horizon.

Q.E.D.

6 Asymptotic completeness of operators \(W_{0, \infty}^{\pm}\) and \(W_{\infty}^{\pm}\) (massless case)

We assume again that \(m = 0\) and \(r_+ = +\infty\). We introduce the inverse wave operators at the horizon and at infinity, defined for \(\Psi_0 \in \mathcal{H}\) by

\[
\tilde{W}_{0, \infty}^{\pm} \Psi_0 = s - \lim_{t \to \pm \infty} e^{-iH_0 t} J_0^* e^{iH_0} \Psi_0 \text{ in } \mathcal{H},
\]

(78)

\[
\tilde{W}_{\infty}^{\pm} \Psi_0 = s - \lim_{t \to \pm \infty} e^{-iH_\infty t} J_\infty^* e^{iH_\infty} \Psi_0 \text{ in } \mathcal{H},
\]

(79)

where \(J_0^*\) and \(J_\infty^*\) are respectively the adjoints of \(J_0\) and \(J_\infty\). We also define the wave operators \(W^+\) and \(W^-\) by

\[
\Psi_0 \in \mathcal{H}_0^{\pm}, \quad \Psi_\infty \in \mathcal{H}_\infty \quad W^\pm(\Psi_0, \Psi_\infty) = W_0^\pm \Psi_0 + W_\infty^\pm \Psi_\infty
\]

(80)

as well as the inverse wave operators \(\tilde{W}^+, \tilde{W}^\pm\)

\[
\Psi_0 \in \mathcal{H} \quad \tilde{W}^\pm \Psi_0 = \left( \tilde{W}_0^\pm \Psi_0, \tilde{W}_\infty^\pm \Psi_\infty \right).
\]

(81)

Eventually, we define the scattering operator

\[
S = \tilde{W}^+ W^-.
\]

(82)

**Theorem 6.1.** Operators \(\tilde{W}_0^{\pm}\) (resp. \(\tilde{W}_\infty^{\pm}\)) are well defined from \(\mathcal{H}\) into \(\mathcal{H}_0^{\pm}\) (resp. from \(\mathcal{H}\) into \(\mathcal{H}_\infty\)), are independent of the choice of \(\chi_0\) (resp. \(\chi_\infty\)) and their norm is lower or equal to 1. Moreover

- \(W^\pm\) is an isometry of \(\mathcal{H}_0^{\pm} \times \mathcal{H}_\infty\) onto \(\mathcal{H}\).
- \(\tilde{W}^\pm\) is an isometry of \(\mathcal{H}_0^{\pm} \times \mathcal{H}_\infty\) onto \(\mathcal{H}_0^{\pm} \times \mathcal{H}_\infty\).
- \(S\) is an isometry of \(\mathcal{H}_0^{-} \times \mathcal{H}_\infty\) onto \(\mathcal{H}_0^{+} \times \mathcal{H}_\infty\).
Proof: For any solution \( \Psi \) of (15) in \( C(\mathbb{R};\mathcal{H}_l), (l, n), \in \mathcal{I}_2 \), we construct asymptotic profiles at the horizon and at infinity. The idea is that each component of \( \Psi \) satisfies an equation of the form

\[
(\partial^2_t - \partial^2_r + V(r_\pm)) f = 0
\]

(83)

where the potential \( V \) has the following properties

\[
\begin{align*}
V &= V_+ - V_-; \quad V_+, V_- \geq 0, \\
V_+(r_*^+) \leq C(1 + |r_*|)^{-1-\varepsilon}, \quad \varepsilon > 0, \\
V_-(-r_*) \leq C(1 + |r_*|)^{-2-\varepsilon}, \quad \varepsilon > 0.
\end{align*}
\]

(84)

We then apply the scattering results of [3]. This suffices to define \( \tilde{\Psi}^\pm \), but to prove the existence of \( \tilde{\Psi}^\pm \), we need to recover a solution of \( (\partial_t - iH_\infty)\Psi = 0 \) from the asymptotic profile at infinity.

Firstly, we study some spectral properties of the operator \( H \):

Proposition 6.1. The point spectrum of \( H \) is empty.

A straightforward consequence of proposition 6.1 is

Corollary 6.1. For \( k \in \mathbb{N} \), the direct sum of the sets

\[
\mathcal{E}_{ln}^k = \{ H^k \Psi; \quad \Psi = (f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_l, \quad 1 \leq i \leq 4 \quad f_i \in C^\infty_0(\mathbb{R}_r) \} \quad (l, n) \in \mathcal{I}_2
\]

(85)

is dense in \( \mathcal{H} \).

Proof of proposition 6.1: Let

\[
\Psi_{ln} = \phi \otimes F_{ln} \in \mathcal{H}_{ln}; \quad \phi = (f_1, f_2, f_3, f_4) \in [L^2(\mathbb{R}, dr^2_\pm)]^4
\]

(86)

such that

\[
H\Psi_{ln} = \lambda \Psi_{ln}; \quad \lambda \in \mathbb{R}.
\]

(87)

Equation (87) is equivalent to

\[
\begin{align*}
f_1' &= -\beta_i f_2 - i\lambda f_3, \\
f_2' &= -\beta_i f_1 + i\lambda f_4, \\
f_3' &= -\beta_i f_4 - i\lambda f_1, \\
f_4' &= -\beta_i f_3 + i\lambda f_2.
\end{align*}
\]

(88)

We first consider the case \( \lambda = 0 \). Putting

\[
\begin{align*}
g_1 &= f_1 + f_2, \\
g_2 &= f_2 - f_1, \\
g_3 &= f_3 + f_4, \\
g_4 &= f_4 - f_3,
\end{align*}
\]

(89)

we see that \( g_1 \) and \( g_3 \) are solutions of

\[
g' = -\beta_i g,
\]

(90)

while \( g_2 \) and \( g_4 \) satisfy

\[
f' = \beta_i f.
\]

(91)

Thus \( \lambda = 0 \) is an eigenvalue for \( H \) if and only if there exists \( l = \frac{1}{2} + k, k \in \mathbb{N} \), such that both equations (90) and (91) have solutions in \( L^2(\mathbb{R}_r, dr^2_\pm). \) \( \beta_i \) being smooth on \( \mathbb{R} \), any solution of (90) or (91) in \( L^1_{1,\infty}(\mathbb{R}) \) is necessarily smooth. Moreover, \( \beta_i \) decreases exponentially when \( r_* \) goes to \( -\infty \), thus

\[
\forall r_*^+ \in \mathbb{R} \quad \beta_i \in L^1([0, r_*^+])
\]

(92)

and both integral equations

\[
f(r_*) = 1 + \int_{-\infty}^{r_*} \beta_i f dr_*,
\]

(93)
\[
g(r_*) = 1 - \int_{-\infty}^{r_*} \beta_t gdr_*
\]

have a unique solution in \(L^\infty (]-\infty, r_*)\), which can be extended on \(\mathbb{R}\) as a smooth but not square integrable function. Therefore, (90) and (91) have no non-trivial solution in \(L^2(\mathbb{R})\) and \(\lambda = 0\) is not an eigenvalue for \(H\).

If now we suppose \(\lambda \neq 0\), the components of \(\phi\) satisfy

\[
\begin{align*}
\phi'' &= (\beta^2 - \lambda^2) \phi_1 - \beta_1 \phi_2, \\
\phi' &= (\beta^2 - \lambda^2) \phi_2 - \beta_2 \phi_1, \\
\phi &= (\beta^2 - \lambda^2) \phi_3 - \beta_3 \phi_4, \\
\phi' &= (\beta^2 - \lambda^2) \phi_4 - \beta_4 \phi_3,
\end{align*}
\]

Functions \(g_1 = f_1 + f_2\) and \(g_3 = f_3 + f_4\) are eigenvectors in \(L^2(\mathbb{R})\) for the operator

\[
L_1 = -\partial^2_{r_*} + \beta^2_1 (r_*) - \beta_1 (r_*)
\]

associated with the eigenvalue \(\lambda^2 > 0\), whereas \(g_2 = f_2 - f_1\) and \(g_4 = f_4 - f_3\) are eigenvectors in \(L^2(\mathbb{R})\) for the operator

\[
L_2 = -\partial^2_{r_*} + \beta^2_2 (r_*) + \beta_2 (r_*)
\]

associated with the eigenvalue \(\lambda^2 > 0\). It is easily seen that potentials

\[
\Psi_1 (r_*) = \beta_1^2 (r_*) - \beta_1 (r_*)
\]

and

\[
\Psi_2 (r_*) = \beta_2^2 (r_*) + \beta_2 (r_*)
\]

satisfy (84). Therefore, the operators \(L_1\) and \(L_2\) are of the same type as the second order operators studied in [3] and have no strictly positive eigenvalue.

\[Q.E.D.\]

**Proof of corollary 6.1:** For \((l, n) \in I_{\frac{1}{2}}^1\) and \(k \in \mathbb{N}\), if

\[
\Psi = \phi \otimes F_{ln} \in \mathcal{H}_{ln} : \phi \in [C_0^\infty (\mathbb{R}_{r_*})]^4,
\]

then \(\Psi\) belongs to \(D (H^k |_{\mathcal{H}_{ln}})\). \(\mathcal{E}_{ln}^k\) is well-defined and is a subset of \(\mathcal{H}_{ln}\). To prove corollary 6.1 it suffices to establish that for \((l, n) \in I_{\frac{1}{2}}^1\) and \(k \in \mathbb{N}\), \(\mathcal{E}_{ln}^k\) is dense in \(\mathcal{H}_{ln}\). Let

\[
\Psi_0 = \phi_0 \otimes F_{ln} \in \mathcal{H}_{ln}
\]

be orthogonal to \(\mathcal{E}_{ln}^k\). Then, for \(\phi \in [C_0^\infty (\mathbb{R}_{r_*})]^4\)

\[
(\phi_0, H^k |_{\mathcal{H}_{ln}} \phi)_{L^2(\mathbb{R}_{r_*})} = 0,
\]

\(H^k |_{\mathcal{H}_{ln}}\) being here considered as an operator on \([L^2(\mathbb{R}_{r_*})]^4\). We have

\[
H^k |_{\mathcal{H}_{ln}} \phi_0 = 0 \quad \text{in} \quad [D'(\mathbb{R}_{r_*})]^4
\]

(100)

where \(D'(\mathbb{R}_{r_*})\) is the space of distributions on \(\mathbb{R}_{r_*}\). From (100), we deduce that \(\Psi_0\) belongs to \(D(H^k |_{\mathcal{H}_{ln}})\) and

\[
H^k \Psi_0 = 0 \quad \text{in} \quad \mathcal{H}_{ln}.
\]

(101)

We know by proposition 6.1 that (101) has no non-trivial solution in \(\mathcal{H}_{ln}\). Thus \(\mathcal{E}_{ln}^k\) is dense in \(\mathcal{H}_{ln}\).

\[Q.E.D.\]
We also study the spectral properties of operators $L_1$, $L_2$. We recall their definition for $l - 1/2 \in \mathbb{N}$

$$i = 1, 2 \quad L_i = -\partial_{r_+}^2 + V_i(r_+) \quad ; \quad V_i(r_+) = \beta_i^2(r_+) + (-1)^i \beta_i'(r_+).$$  

(102)

**Proposition 6.2.** For $l - 1/2 \in \mathbb{N}$, the spectrum of operators $L_1$ and $L_2$ is purely absolutely continuous.

**Proof:** We already know that potentials $V_1$ and $V_2$ satisfy (84), which, from [3] implies that the singular spectrum of $L_1$ and $L_2$ is empty, that their absolutely continuous spectrum is $[0, +\infty]$ and that their point spectrum contains at the most a finite number of negative or zero eigenvalues, all of them being simple. Furthermore, $V_1$ and $V_2$ decrease exponentially when $r_* \to -\infty$ and 0 is not an eigenvalue. We show that $L_1$ and $L_2$ do not have any strictly negative eigenvalue either by a method similar to the one used in [3]. We recall that for $l - 1/2 \in \mathbb{N}$, equations

$$1 \leq i \leq 2 \quad L_i f = 0$$

(103)

both have on $\mathbb{R}_{r_*}$ a unique continuous strictly positive solution, given respectively by (93) and (94). We consider the general case of a potential

$$V \in L^\infty(\mathbb{R}_{r_*}) \cap L^2(\mathbb{R}_{r_*})$$

(104)

such that there exists a function $g$, continuous and strictly positive on $\mathbb{R}_{r_*}$, satisfying

$$L_V g = 0 \quad ; \quad L_V = -\partial_{r_+}^2 + V.$$

(105)

Let $f \in L^2(\mathbb{R}_{r_*})$ be such that

$$L_V f = -\lambda f \quad , \quad \lambda > 0,$$

(106)

which implies

$$f \in H^2(\mathbb{R}_{r_*}).$$

(107)

We define the cut-off function

$$\chi \in C_0^\infty(\mathbb{R}_{r_*}) \quad , \quad \text{for } |r_*| \leq \frac{1}{2} \quad \chi(r_*) = 1 \quad , \quad \text{for } |r_*| \geq 1 \quad \chi(r_*) = 0.$$  

(108)

Putting for $n \geq 1$

$$f_n(r_*) = \chi\left(\frac{r_*}{n}\right) f(r_*),$$

(109)

we easily see that

$$\int_{[-n,n]} \left(|f_n'|^2 + V |f_n|^2\right) dr_* = -\lambda \int_{[-\frac{n}{2}, \frac{n}{2}]} |f|^2 dr_* + o(1).$$

(110)

Thus, for $n$ large enough

$$\int_{[-n,n]} \left(|f_n'|^2 + V |f_n|^2\right) dr_* < 0.$$

The operator $-\partial_{r_+}^2 + V$ on $L^2([-n,n])$ with domain $\{y \in H^2([-n,n]); \ y(\pm n) = 0\}$ has a strictly negative eigenvalue $-\lambda_n$ associated with an eigenvector $u$

$$\begin{cases} 
-u'' + Vu = -\lambda_n u \quad ; \quad -n < r_* < n, \\
u(-n) = u(n) = 0.
\end{cases}$$

(111)

Even if it means changing $u$ into $-u$, there exist $\alpha$ and $\beta$ such that

$$-n \leq \alpha < \beta \leq n,$$

$$u(\alpha) = u(\beta) = 0 \quad , \quad u'(\alpha) > 0 \quad , \quad u'(\beta) < 0,$$

(112)

$$u > 0 \quad \text{for } \alpha < r_* < \beta.$$

We denote

$$I = \int_{\alpha}^{\beta} (u' g - u g')' dr_*.$$
On the one hand, we can write
\[ I = u'(\beta) g(\beta) - u'(\alpha) g(\alpha), \]
g being strictly positive on \( \mathbb{R} \), (112) yields
\[ I < 0. \]
On the other hand
\[ (u'g - ug')' = u'' g - g'' u = -\lambda u g, \]
thus
\[ I = \lambda \int_\alpha^\beta u g \, dr > 0. \]
We end up with a contradiction, which means that \( L_V \) has no strictly negative eigenvalue.

Q.E.D.

We now prove the existence of the inverse wave operators \( \tilde{W}^+_0 \) and \( \tilde{W}^-_\infty \). For \( (l,n) \in \mathcal{I}_{\frac{1}{2}} \), we consider the orthogonal decomposition of \( H_{\ln} \)
\[ \mathcal{H}_{\ln} = \mathcal{H}_{\ln}^+ \oplus \mathcal{H}_{\ln}^- , \quad \mathcal{H}_{\ln}^+ = \{ \Psi = t (f_1, f_2, f_3, f_4) \otimes F_{\ln} \in \mathcal{H}_{\ln} ; \quad f_2 = \mp f_1 , \quad f_4 = \pm f_3 \} . \] (113)
Each \( \mathcal{H}_{\ln}^\pm \) is stable under \( H \) and by corollary 6.1, for \( (l,n) \in \mathcal{I}_{\frac{1}{2}} \), \( k \in \mathbb{N} \), the sets
\[ \mathcal{E}_{\ln}^{k,\pm} = \mathcal{E}_{\ln}^k \cap \mathcal{H}_{\ln}^\pm = \{ H^k \Psi ; \quad \Psi = \Psi_{\ln} \otimes F_{\ln} \in \mathcal{H}_{\ln}^\pm ; \quad f_1, f_3 \in C_0^\infty (\mathbb{R}_r) \} \] (114)
are respectively dense in \( \mathcal{H}_{\ln}^+ \) and \( \mathcal{H}_{\ln}^- \). For \( \Psi_0 \in \mathcal{E}_{\ln}^{2,\pm} \) we establish the existence of the strong limits (78) and (79) defining \( \tilde{W}^+_0 \Psi_0 \) and \( \tilde{W}^-_\infty \Psi_0 \). The following lemma guarantees the existence of asymptotic profiles for \( \Psi_0 \). The details of its proof will be given after the proof of theorem 6.1.

Lemma 6.1. Given \( \Psi_0 \in \mathcal{E}_{\ln}^{2,\varepsilon} \), \( (l,n) \in I_{\frac{1}{2}} \), there exists
\[ \Psi_1 \in \left[ C (\mathbb{R}_t; H^1 (\mathbb{R}_r)) \cap C^1 (\mathbb{R}_t; L^2 (\mathbb{R}_r)) \right] \right]^4 \otimes F_{\ln} \] (115)
such that
\[ \partial_t \Psi_1 = i H_0 \Psi_1 , \] (116)
and
\[ \lim_{t \to +\infty} \| e^{i \varepsilon H t} \Psi_0 - \Psi_1 (t) \|_{\mathcal{H}} = 0. \] (117)
Any solution of (116) in \( C (\mathbb{R}_t; \mathcal{H}) \) and in particular \( \Psi_1 \) can be expressed in the form
\[ \Psi_1 (t) = e^{i H_0 t} \Psi_0^+ + e^{i H_0 t} \Psi_0^- \] (118)
where
\[ \Psi_0^+ \in \mathcal{H}_{\ln}^+ , \quad \Psi_0^- \in \mathcal{H}_{\ln}^- . \] (119)
Thus, for a cut-off function \( \chi_0 \) satisfying (44), we have
\[ \lim_{t \to +\infty} \| J_0 \Psi_1 (t) - e^{i H_0 t} \Psi_0^+ \|_{\mathcal{H}} = 0. \] (120)
That is to say that for \( \Psi_0 \in \mathcal{E}_{\ln}^{2,\varepsilon} \), \( (l,n) \in I_{\frac{1}{2}} \), \( \varepsilon = +, - \), there exists
\[ \Psi_0^+ \in \mathcal{H}_{\ln}^+ \cap \mathcal{H}_{\ln}^\varepsilon \] (121)
such that
\[ \lim_{t \to +\infty} \| J_0 e^{i H t} \Psi_0 - e^{i H_0 t} \Psi_0^+ \|_{\mathcal{H}} = 0. \] (122)
and of course, we can similarly prove the existence of
\[ \Psi_0^\pm \in \mathcal{H}_0 \cap \mathcal{H}_{ln}^\varepsilon \] (123)
such that
\[ \lim_{t \to -\infty} \| J_0 e^{iHt} \psi_0 - e^{iH_0 t} \psi_0 \|_{\mathcal{H}} = 0. \] (124)

From (121) to (124), we conclude that \( \tilde{W}_0^\pm \psi_0 \) is well-defined for \( \psi_0 \in \mathcal{E}_{ln}^\varepsilon, (l, n) \in \mathcal{I}_l, \varepsilon = +, - \), and
\[ \tilde{W}_0^\pm \psi_0 \in \mathcal{H}_0^\pm, \quad \| \tilde{W}_0^\pm \psi_0 \|_{\mathcal{H}_0} \leq \| \psi_0 \|_{\mathcal{H}}. \] (125)

Then, corollary 6.1 yields that the operator \( \tilde{W}_\infty^\pm \) is well-defined for \( \psi_0 \in \mathcal{H}_0^\pm \) and for \( \psi_0 \in \mathcal{H}_0^\pm \)
\[ \tilde{W}_\infty^\pm \psi_0 = 0. \] (126)

**Lemma 6.2.** The operator \( W_0^\infty \)
\[ W_0^\infty \psi_0 = s - \lim_{t \to +\infty} e^{-iH_\infty t} J_\infty^* e^{iH_0 t} \psi_0 \] (127)
is well-defined from \( \mathcal{H}_0 \) to \( \mathcal{H}_\infty \) and is independent of the choice of \( \chi_\infty \) satisfying (65). Of course \( W_0^\infty \) is defined as well from \( \mathcal{H}_0^\pm \) to \( \mathcal{H}_\infty \) and for \( \psi_0 \in \mathcal{H}_0^\pm \)
\[ W_0^\infty \psi_0 = 0. \]

Lemma 6.2, and (118), (119) yield the existence of
\[ \Psi_\infty^\pm \in \mathcal{H}_\infty \] (128)
such that
\[ \lim_{t \to +\infty} \| J_\infty^* \psi_1(t) - e^{iH_\infty t} \psi_\infty^\pm \|_{\mathcal{H}_\infty} = 0 \] (129)
and therefore
\[ \lim_{t \to +\infty} \| J_\infty^* e^{iHt} \psi_0 - e^{iH_\infty t} \psi_\infty^\pm \|_{\mathcal{H}_\infty} = 0. \] (130)
which enables us to define \( \tilde{W}_\infty^\pm \) on \( \mathcal{E}_{ln}^\varepsilon, (l, n) \in \mathcal{I}_l \) and by density on \( \mathcal{H} \). The same thing can be done for \( \tilde{W}_\infty^\pm \). Let \( \chi_\infty \) and \( \chi_\infty' \) be two cut-off functions satisfying (65) and \( J_\infty \) and \( J_\infty' \) the associated identifying operators. For \( t \in \mathbb{R}, \psi_0 \in \mathcal{H} \)
\[ \| e^{-iH_\infty t} J_\infty^* e^{iHt} \psi_0 - e^{-iH_\infty t} J_\infty' e^{iHt} \psi_0 \|_{\mathcal{H}_\infty} \leq \| (\chi_\infty - \chi_\infty') e^{iHt} \psi_0 \|_{\mathcal{H}}, \]
and
\[ \lim_{t \to \pm\infty} \| e^{-iH_\infty t} J_\infty^* e^{iHt} \psi_0 - e^{-iH_\infty t} J_\infty' e^{iHt} \psi_0 \|_{\mathcal{H}_\infty} = 0. \]
Thus, the operators \( \tilde{W}_\infty^\pm \) are independent of the choice of \( \chi_\infty \) and by a similar argument, \( \tilde{W}_0^\pm \) are independent of the choice of \( \chi_0 \).

We still have to prove that \( W^\pm \) and \( \tilde{W}^\pm \) are bijective isometries, which yields that \( S \) is a bijective isometry by construction. Let \( \psi \in \mathcal{H} \) and
\[ \psi_0^\pm = \tilde{W}_0^\pm \psi, \quad \psi_\infty^\pm = \tilde{W}_\infty^\pm \psi. \] (131)
For $\chi_o$ satisfying (44) and $\chi_\infty$ satisfying (65), we have

$$\lim_{t \to \pm \infty} \| J_0 (e^{iHt}\Psi - e^{iH_0t}\Psi_0^\pm) \|_{\mathcal{H}} = 0,$$

(132)

$$\lim_{t \to \pm \infty} \| J_\infty J_\infty^* e^{iHt}\Psi - J_\infty e^{iH_\infty t}\Psi_\infty^\pm \|_{\mathcal{H}} = 0,$$

(133)

$J_\infty J_\infty^*$ being simply the multiplication by $\chi_\infty$. The local energy of $e^{iHt}\Psi$ goes to 0 when $t$ goes to $\pm \infty$, therefore

$$\lim_{t \to \pm \infty} \| (\chi_o + \chi_\infty - 1) e^{iHt}\Psi \|_{\mathcal{H}} = 0.$$

(134)

(132), (133) and (134) imply

$$\lim_{t \to \pm \infty} \| J_0^* e^{iHt}\Psi - J_0 e^{iH_0t}\Psi_0^\pm - J_\infty e^{iH_\infty t}\Psi_\infty^\pm \|_{\mathcal{H}} = 0,$$

(135)

which means

$$W^\pm \tilde{W}^\pm = \mathbb{I}_{\mathcal{H}}.$$

(136)

If on the other hand we consider

$$\Psi_0^\pm \in \mathcal{H}_0^\pm, \quad \Psi_\infty \in \mathcal{H}_\infty,$$

(137)

and put

$$\Psi = W^\pm (\Psi_0^+, \Psi_\infty^-),$$

(138)

we have (135) from which we get

$$\lim_{t \to \pm \infty} \| J_0^* (e^{iHt}\Psi - J_0 e^{iH_0t}\Psi_0^\pm - J_\infty e^{iH_\infty t}\Psi_\infty^\pm) \|_{\mathcal{H}} = 0$$

(139)

and

$$\lim_{t \to \pm \infty} \| J_\infty^* (e^{iHt}\Psi - J_0 e^{iH_0t}\Psi_0^\pm - J_\infty e^{iH_\infty t}\Psi_\infty^\pm) \|_{\mathcal{H}_\infty} = 0.$$

(140)

The local energy of $e^{iH_0t}\Psi_0^\pm$ and $e^{iH_\infty t}\Psi_\infty^\pm$ goes to 0 when $|t|$ goes to $+\infty$, therefore (139) and (140) yield

$$\lim_{t \to \pm \infty} \| J_0^* e^{iHt}\Psi - e^{iH_0t}\Psi_0^\pm \|_{\mathcal{H}} = 0$$

(141)

and

$$\lim_{t \to \pm \infty} \| J_\infty^* e^{iHt}\Psi - e^{iH_\infty t}\Psi_\infty^\pm \|_{\mathcal{H}_\infty} = 0,$$

(142)

thus

$$\tilde{W}^\pm W^\pm = \mathbb{I}_{\mathcal{H}_0^\pm \times \mathcal{H}_\infty}.$$

(143)

(136) and (143) show that $W^\pm$ and $\tilde{W}^\pm$ are all bijections and if we choose $\chi_o$ and $\chi_\infty$ such that their supports have no intersection, we deduce from (135)

$$\| \Psi \|_{\mathcal{H}} = \| \Psi_0^\pm \|_{\mathcal{H}_0^\pm} + \| \Psi_\infty^\pm \|_{\mathcal{H}_\infty}.$$

(144)

Q.E.D.

Proof of lemma 6.1: Let $\Psi_0 \in \mathcal{E}_l^{2\epsilon}, (l, n) \in \mathcal{I}_{2\epsilon}, \epsilon = +, -$. There exists

$$\Psi_0' = t (f_1, -\epsilon f_3, \epsilon f_3) \otimes F_l \in \mathcal{E}_l^{2\epsilon}$$

(145)

such that

$$\Psi_0 = iH \Psi_0'$$

(146)
\[ \Psi_0'' = i (g_1, -\varepsilon g_1, g_3, \varepsilon g_3) \otimes F_{ln} \in E_{ln}^{0\varepsilon} \] (147)

such that
\[ \Psi_0' = -i H \Psi_0''. \] (148)

We denote
\[ \tilde{\Psi} = e^{iHt} \Psi_0' ; \quad \tilde{\Psi} = \phi \otimes F_{ln} = i (\phi_1, -\varepsilon \phi_1, \phi_3, \varepsilon \phi_3) \otimes F_{ln} \] (149)

and
\[ \Psi = \partial_t \tilde{\Psi} = i H \tilde{\Psi}. \] (150)

On the one hand, applying \( \partial_t + iH \) to equation
\[ (\partial_t - iH) \tilde{\Psi} = 0, \]
we obtain
\[ (\partial_t^2 - H^2) \tilde{\Psi} = 0 \]
which, taking into account the fact that \( \tilde{\Psi} \) takes its values in \( \mathcal{H}_{ln} \) can also be written
\[ (\partial_t^2 - \partial_r^2 + \beta_1^2 + \varepsilon \beta_1') \phi_1 = 0, \] (151)
\[ (\partial_t^2 - \partial_r^2 + \beta_2^2 - \varepsilon \beta_2') \phi_3 = 0. \] (152)

On the other hand
\[ \phi_1 |_{t=0} = f_1 ; \quad \phi_3 |_{t=0} = f_3 ; \quad f_1, f_3 \in C_0^\infty (\mathbb{R}_r) \] (153)

and since \( \Psi_0 = H^2 \Psi_0'' \)
\[ \partial_t \phi_1 |_{t=0} = (-\partial_r^2 + \beta_1^2 + \varepsilon \beta_1') g_1 , \quad g_1 \in C_0^\infty (\mathbb{R}_r) \] (154)
\[ \partial_t \phi_3 |_{t=0} = (-\partial_r^2 + \beta_2^2 - \varepsilon \beta_2') g_3 , \quad g_3 \in C_0^\infty (\mathbb{R}_r). \] (155)

The scattering results obtained in [3] together with proposition 6.2 imply that for any solution
\[ f \in C (\mathbb{R}_t; H^1 (\mathbb{R}_r)) \cap C^1 (\mathbb{R}_t; L^2 (\mathbb{R}_r)) \]
of equation
\[ (\partial_t^2 - \partial_r^2 + \beta_i^2 + \eta \beta_i') f = 0 , \quad \eta = +, - \]
with initial data
\[ f |_{t=0} = \mu_1 , \quad \partial_t f |_{t=0} = (-\partial_r^2 + \beta_i^2 + \eta \beta_i') \mu_2 \]
such that
\[ i = 1, 2 \quad \mu_i \in L^2 (\mathbb{R}_r) ; \quad (-\partial_r^2 + \beta_i^2 + \eta \beta_i') \mu_i \in L^2 (\mathbb{R}_r), \]
there exists a solution
\[ f_1 \in C (\mathbb{R}_t; H^1 (\mathbb{R}_r)) \cap C^1 (\mathbb{R}_t; L^2 (\mathbb{R}_r)) \] (156)
of
\[ (\partial_t^2 - \partial_r^2) f_1 = 0 \] (157)
such that
\[ \lim_{t \to +\infty} \| f(t) - f_1(t) \|_{H^1 (\mathbb{R}_r)} + \| \partial_t f(t) - \partial_t f_1(t) \|_{L^2 (\mathbb{R}_r)}. \]

\( \tilde{\Psi} \) is the solution of (15) with initial data
\[ \Psi_0' \in [C_0^\infty (\mathbb{R}_r)]^4 \otimes F_{ln} \]
therefore in particular,
\[ \phi_1, \phi_2 \in C (\mathbb{R}_t; H^1 (\mathbb{R}_r)) \cap C^1 (\mathbb{R}_t; L^2 (\mathbb{R}_r)) \]
and (151) to (155) yield the existence of
\[ \tilde{\Psi}_1 \in \left[ C (\mathbb{R}_t; H^1 (\mathbb{R}_r)) \cap C^1 (\mathbb{R}_t; L^2 (\mathbb{R}_r)) \right]^4 \otimes F_{ln} \]
such that
\[ (\partial_t^2 - \partial_{r_\ast}^2) \tilde{\Psi}_1 = 0 \]
and
\[ \lim_{t \to +\infty} \left\| e^{iHt} \tilde{\Psi}_0 - \tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0, \quad \lim_{t \to +\infty} \left\| \partial_{r_\ast} \left( e^{iHt} \tilde{\Psi}_0 - \tilde{\Psi}_1 \right) \right\|_{\mathcal{H}} = 0 \]
from which we deduce
\[ \lim_{t \to +\infty} \left\| e^{iHt} \tilde{\Psi}_0 - \partial_t \tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0. \quad (158) \]
\( \Psi_0 \) being an element of \( \mathcal{E}^{2r}_{ln} \subset \mathcal{E}^1_{ln} \), we can apply the previous construction to \( \Psi_0 \). We find that there exists
\[ \Psi_1 \in \left[ \mathcal{C} \left( \mathbb{R}_t; H^1(\mathbb{R}_r^\ast) \right) \cap \mathcal{C}^1 \left( \mathbb{R}_t; L^2(\mathbb{R}_r^\ast) \right) \right] \otimes F_{ln} \]
solution of
\[ (\partial_t^2 - \partial_{r_\ast}^2) \Psi_1 = 0 \]
such that
\[ \lim_{t \to +\infty} \left\| e^{iHt} \Psi_0 - \Psi_1 \right\|_{\mathcal{H}} = 0, \quad \lim_{t \to +\infty} \left\| \partial_{r_\ast} \left( e^{iHt} \Psi_0 - \Psi_1 \right) \right\|_{\mathcal{H}} = 0, \quad (159) \]
\[ \lim_{t \to +\infty} \left\| \partial_t \left( e^{iHt} \Psi_0 - \Psi_1 \right) \right\|_{\mathcal{H}} = 0. \quad (160) \]
From (159) and (160) we deduce
\[ \lim_{t \to +\infty} \left\| (\partial_t - iH_0) \left( e^{iHt} \Psi_0 - \Psi_1 \right) \right\|_{\mathcal{H}} = 0. \quad (161) \]
e\( e^{iHt} \Psi_0 \) being a solution of (15) in \( \mathcal{C} \left( \mathbb{R}_t; \mathcal{H}_{ln} \right) \), we have
\[ (\partial_t - iH) e^{iHt} \Psi_0 = (\partial_t - iH_0 - i\beta_l M) e^{iHt} \Psi_0 = 0 \quad (162) \]
and by (158)
\[ \lim_{t \to +\infty} \left\| i\beta_l M \left( e^{iHt} \Psi_0 - \partial_t \tilde{\Psi}_1 \right) \right\|_{\mathcal{H}} = 0. \]
\( \partial_t \tilde{\Psi}_1 \) is identically zero in
\[ \{(t,r_\ast,\omega); |r_\ast| \leq |t| - R, \omega \in S^2\}, \]
which is not true in general for \( \tilde{\Psi}_1 \), therefore
\[ \lim_{t \to +\infty} \left\| i\beta_l M \partial_t \tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0 \]
and
\[ \lim_{t \to +\infty} \left\| i\beta_l Me^{iHt} \Psi_0 \right\|_{\mathcal{H}} = 0. \quad (163) \]
(161), (162) and (163) give
\[
\lim_{t \to +\infty} \| (\partial_t - iH_0) \Psi_1 \|_{\mathcal{H}} = 0
\]
and \((\partial_t - iH_0) \Psi_1\) being an element of \(\mathcal{C} (\mathbb{R}_t; \mathcal{H})\) and satisfying
\[
(\partial_t + iH_0) [(\partial_t - iH_0) \Psi_1] = 0
\]
we must have
\[
(\partial_t - iH_0) \Psi_1 = 0.
\]
Q.E.D.

Proof of lemma 6.2: Let
\[
\Psi_0 \in \mathcal{H}_0 \cap \mathcal{C}^{0\varepsilon} \quad (l, n) \in \mathcal{I}_{1/2} \quad \varepsilon = +, -
\]
with
\[
\text{Supp}(\Psi_0) \subset [-R, R]_r \times S^2_{\theta, \phi} \quad R > 0.
\]
\(\Psi_0\) can be written
\[
\Psi_0 = \imath (f_0, -\varepsilon f_0, f_0, \varepsilon f_0) \otimes F_{ln} \quad f_0 \in \mathcal{C}^{0\varepsilon}_0 (\mathbb{R}_r) \quad \text{Supp} f_0 \subset [-R, R]
\]
and
\[
\varepsilon^{iH_0} \Psi_0 = \imath (f, -\varepsilon f, f, \varepsilon f) \otimes F_{ln} \quad f(t, r_*) = f_0(r_* - t).
\]
\(f\) is the solution of
\[
(\partial^2_t - \partial^2_{r_*}) f = 0
\]
associated with the initial data
\[
f|_{t=0} = f_0 \quad \partial_t f|_{t=0} = -\partial_{r_*} f_0.
\]
Instead of applying Cook’s method to operators \(H_\infty\) and \(H_0\), which would give an apparently long-range perturbation at infinity, we work on the second order scalar equations and establish the existence of \(g_\eta\) solution of
\[
\left\{ \begin{array}{l}
(\partial^2_t - \partial^2_{r_*} + V_\eta(r_*)) g_\eta = 0 \\
V_\eta(r_*) = \chi_\infty(r_*) \frac{l}{2} \left( (l + \frac{1}{2})^2 + \eta (l + \frac{1}{2}) \right), \quad \eta = +, -, 
\end{array} \right.
\]
where \(\chi_\infty\) is a cut-off function satisfying (65); the solution \(g_\eta\) being such that
\[
\lim_{t \to +\infty} \| \partial_t (g_\eta - f) \|_{L^2(\mathbb{R})} = 0, \quad \lim_{t \to +\infty} \| \partial_{r_*} (g_\eta - f) \|_{L^2(\mathbb{R})} = 0,
\]
\[
\lim_{t \to +\infty} \frac{1 + \frac{3}{2}}{r} (g_\eta - f) \|_{L^2(\mathbb{R})} = 0.
\]
In the case where \(l = 1/2\) and \(\eta = -, \) equations (168) and (170) are the same and it suffices to take \(g_- = f\).

Let us now assume
\[
\left( l + \frac{1}{2} \right)^2 + \eta \left( l + \frac{1}{2} \right) > 0.
\]
We write equations (168) and (170) in their hamiltonian form
\[
\partial_t \left( \frac{f}{\partial_t f} \right) = -\left( \begin{array}{cc} 0 & -1 \\
-\partial^2_{r_*} & 0 \end{array} \right) \left( \frac{f}{\partial_t f} \right) = -A_0 \left( \frac{f}{\partial_t f} \right),
\]
\[
\partial_t \left( \frac{g}{\partial_t g} \right) = -\left( \begin{array}{cc} 0 & -1 \\
-\partial^2_{r_*} + V_\eta & 0 \end{array} \right) \left( \frac{g}{\partial_t g} \right) = -A_\eta \left( \frac{g}{\partial_t g} \right).
\]
The operator $iA_0$ is skew-adjoint with dense domain on
\[ H_0 = BL^1(\mathbb{R}_r) \times L^2(\mathbb{R}_r) \] (176)
completion of $[C^0_0(\mathbb{R}_r)]^2$ for the norm
\[ \| \mathcal{I} (f_1, f_2) \|_{H_0}^2 = \int_\mathbb{R} \left\{ |\partial_r f_1|^2 + |f_2|^2 \right\} \, dr. \] (177)
and $iA_0$ is skew-adjoint with dense domain (cf. [3]) on
\[ H = H_1 \times L^2(\mathbb{R}_r) \] (178)
completion of $[C^0_0(\mathbb{R}_r)]^2$ for the norm
\[ \| \mathcal{I} (g_1, g_2) \|_{H}^2 = \int_\mathbb{R} \left\{ |\partial_r g_1|^2 + |g_2|^2 + \nu_0 |g_1|^2 \right\} \, dr. \] (179)
Under assumption (173), the norm (179) is equivalent to
\[ \| \mathcal{I} (g_1, g_2) \|_{H_0}^2 = \int_\mathbb{R} \left\{ \frac{(t + \frac{1}{2}) \chi_\infty}{r_*} g_1 \right\}^2 \, dr \] (180)
Moreover, any solution $\mathcal{I} (g, \partial t g) \in C(\mathbb{R}_r; H)$ of (170) satisfies the following energy estimate: for $r_*^1 < r_*^2$
and $t \in \mathbb{R}$
\[ \int_{r_*^1 < r_* < r_*^2} \left\{ |\partial_r g(t)|^2 + |\partial t g(t)|^2 + \nu_0 |g(t)|^2 \right\} \, dr \leq \int_{r_*^1 - |t| < r_* < r_*^2 + |t|} \left\{ |\partial_r g(0)|^2 + |\partial t g(0)|^2 + \nu_0 |g(0)|^2 \right\} \, dr \] (181)
which is very easily obtained by multiplying (170) by $\partial t g$ and integrating by parts on the domain
\[ \Omega_{t, r_*^1, r_*^2} = \{ (\tau, r_*); \ \tau \in (0, t), \ r_*^1 - |t - \tau| < r_* < r_*^2 + |t - \tau| \} \] (182)
f0 being in $C^0_0(\mathbb{R}_r)$, we can consider that
\[ e^{-A_0 t} \mathcal{I} (f_0, -\partial_r f_0) \in C(\mathbb{R}_r; H) \]
and we apply Cook’s method to prove the existence in $H$ of the limit
\[ \left( \begin{array}{c} g_0 \ 
\phi_0 = \mathcal{I} (f_0, -\partial_r f_0), \ \phi_\infty = \mathcal{I} (g_0, g_1) \end{array} \right) \] (183)
We shall denote
\[ \phi_0 = \mathcal{I} (f_0, -\partial_r f_0), \ \phi_\infty = \mathcal{I} (g_0, g_1). \] (184)
We have
\[ \| \partial_t (e^{A_0 t} e^{-A_0 t} \phi_0) \|_H = \| (A_\eta - A_0) e^{-A_0 t} \phi_0 \|_H = \| V_\eta (r_*) f_0 (r_* - t) \|_{L^2(\mathbb{R}_r)} \leq \| f_0 \|_{L^2(\mathbb{R}_r)} \| V_\eta \|_{L^\infty(\mathbb{R}_r > t - R)} \] and for $r_*$ large enough
\[ V_\eta (r_*) = C r_*^{-2}, \ C > 0, \] (185)
thus
\[ \| \partial_t (e^{A_0 t} e^{-A_0 t} \phi_0) \|_H = O(t^{-2}) ; \ t \to +\infty, \] and
\[ \| \partial_t (e^{A_0 t} e^{-A_0 t} \phi_0) \|_H \in L^1(t > 0). \]
The limit (183) is therefore well-defined and if $g_\eta$ is the solution of (170) such that
\[ \left( \begin{array}{c} g_\eta (t) \ 
\frac{\partial g_\eta (t)}{\partial t} \end{array} \right) = e^{-A_0 t} \phi_\infty, \] (186)
then
\[
\lim_{t \to +\infty} \| t (g_\eta, \partial_t g_\eta) - t (f, \partial_t f) \|_H = 0. \tag{187}
\]

This last limit together with the equivalence of norms (179) and (180) gives (171) and (172). Moreover, for \( r_* < t - R \)
\[
g_\eta(t, r_*) = 0 \quad \text{and} \quad \partial_t g_\eta(t, r_*) = 0. \tag{188}
\]
Indeed, for \( t \in \mathbb{R}, \varepsilon > 0 \) we choose \( \tau \in \mathbb{R} \) such that
\[
\| \phi_\infty - e^{iA_\tau} e^{-iA_\tau} \phi_0 \|_H \leq \varepsilon, \quad \tau \geq t. \tag{189}
\]
For \( \langle f_1, f_2 \rangle \in \mathbb{H} \), we denote
\[
\mathcal{L} (\langle f_1, f_2 \rangle) = |\partial_{r_*} f_1|^2 + V_0 |f_1|^2 + |f_2|^2. \tag{190}
\]
Let us consider
\[
\int_{r_* < t - R} \mathcal{L} (e^{-iA_{\tau t}} \phi_\infty) \, dr_* \leq \int_{r_* < t - R} \mathcal{L} [e^{-iA_{\tau t}} (\phi_\infty - e^{iA_\tau} e^{-iA_\tau} \phi_0)] \, dr_*
+ \int_{r_* < t - R} \mathcal{L} (e^{-iA_{\tau (t-\tau)}} e^{-iA_\tau} \phi_0) \, dr_*,
\]
(181) and (189) yield
\[
\int_{r_* < t - R} \mathcal{L} (e^{-iA_{\tau t}} \phi_\infty) \, dr_* \leq \varepsilon^2 + \int_{r_* < t - \tau} \mathcal{L} (e^{-iA_{\tau} \phi_0}) \, dr_*,
\]
and this last integral is zero since
\[
\text{Supp} (e^{-iA_{\tau} \phi_0}) \subset [\tau - R, \tau + R].
\]
(188) is therefore satisfied and for \( t \) large enough \( g_\eta \) is a solution of
\[
\left[ \partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left( \left( l + \frac{1}{2} \right)^2 + \eta \left( l + \frac{1}{2} \right) \right) \right] g_\eta = 0. \tag{191}
\]
Let us now introduce
\[
\Psi_\infty(t) = \langle t (g_{-\varepsilon}(t), -\varepsilon g_{-\varepsilon}(t), g_{\varepsilon}(t), \varepsilon g_{\varepsilon}(t)) \rangle \otimes F_0.
\]
There exists \( t_0 > 0 \) such that, for \( t \geq t_0 \), \( g_{\varepsilon} \) and \( g_{-\varepsilon} \) satisfy
\[
\left[ \partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left( \left( l + \frac{1}{2} \right)^2 + \varepsilon \left( l + \frac{1}{2} \right) \right) \right] g_{\varepsilon} = 0, \tag{192}
\]
\[
\left[ \partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left( \left( l + \frac{1}{2} \right)^2 - \varepsilon \left( l + \frac{1}{2} \right) \right) \right] g_{-\varepsilon} = 0 \tag{193}
\]
with
\[
g_{\varepsilon}, g_{-\varepsilon} \in C ([t_0, +\infty[; \mathbb{H}_1) \quad \text{and} \quad \partial_t g_{\varepsilon}, \partial_t g_{-\varepsilon} \in C ([t_0, +\infty[; L^2(\mathbb{R}_{r_*})]. \tag{194}
\]
Moreover, for \( t \geq t_0 \)
\[
\text{Supp} (g_{\varepsilon}(t), g_{-\varepsilon}(t), \partial_t g_{\varepsilon}(t), \partial_t g_{-\varepsilon}(t)) \subset [t - R, +\infty[ \subset [0, +\infty[. \tag{195}
\]
Thus, the quantities
\[
\partial_t \Psi_\infty, \partial_{r_*} \Psi_\infty, \left( l + \frac{1}{2} \right) r_*^{-1} \Psi_\infty
\]

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belong to \( C([t_0, +\infty[; \mathcal{H}) \) and (171), (172) yield
\[
\lim_{t \to +\infty} \| \partial_t \left( \tilde{\Psi}_\infty(t) - e^{i H_0 t} \Psi_0 \right) \|_{\mathcal{H}} = 0, \quad \lim_{t \to +\infty} \| \partial_{r_*} \left( \tilde{\Psi}_\infty(t) - e^{i H_0 t} \Psi_0 \right) \|_{\mathcal{H}} = 0, \quad (197)
\]

\[
\lim_{t \to +\infty} \left\| \left( l + \frac{1}{2} \right) r_*^{-1} \tilde{\Psi}_\infty(t) \right\|_{\mathcal{H}} = 0. \quad (198)
\]

In particular, we have
\[
\lim_{t \to +\infty} \left\| \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right\|_{\mathcal{H}} = 0. \quad (199)
\]

Since \( e^{i H_0 t} \Psi_0 \) is a solution of
\[
(\partial_t + L \partial_{r_*}) e^{i H_0 t} \Psi_0 = 0,
\]
we have
\[
\left\| \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right\|_{\mathcal{H}} = 0 \quad t \to +\infty
\]

and therefore
\[
\lim_{t \to +\infty} \left\| \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right\|_{\mathcal{H}} = 0. \quad (200)
\]

We introduce
\[
\Psi_\infty = \tilde{\Psi}_\infty |_{\{r_* \geq 0\}}. \quad (201)
\]

The quantities
\[
\partial_r \Psi_\infty , \quad \partial_{r_*} \Psi_\infty , \quad \left( l + \frac{1}{2} \right) r_*^{-1} \Psi_\infty
\]

belong to \( C([t_0, +\infty[; \mathcal{H}_{\infty}^{\text{lin}}) \) where, for \((l, n) \in I_{\frac{1}{2}}\) and \(\varepsilon = +, -\)
\[
\mathcal{H}_{\infty}^{\text{lin}} = \{ \Psi, f, g, h, f_n \in \mathcal{H}_\infty \}. \quad (202)
\]

From (200), we get
\[
\lim_{t \to +\infty} \left\| \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right\|_{\mathcal{H}_\infty} = 0 \quad (203)
\]

and, the function
\[
\left( \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty \in C([t_0, +\infty[; \mathcal{H}_{\infty}^{\text{lin}})
\]

satisfies
\[
\left( \partial_t - L \partial_{r_*} + i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \left[ \left( \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty \right] = 0. \quad (204)
\]

Therefore, we must have for \( t \geq t_0 \)
\[
\left( \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty(t) = 0 \quad \text{in } \mathcal{H}_\infty.
\]

\( H_1 \) being a distribution space, we can write in the sense of distributions for \( t \geq t_0 \)
\[
\partial_t \left( \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty(t) = \left( \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \partial_t \Psi_\infty(t) = 0 \quad \text{in } \mathcal{H}_\infty,
\]

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which implies that $\partial_t \Psi_\infty$ is a solution in $\mathcal{C}([t_0, +\infty[; \mathcal{H}^{cl_\infty}_{\infty})$ of

$$(\partial_t - iH_{\infty}) \Psi = 0.$$  

This solution can be extended to $\mathcal{C}(\mathbb{R}; \mathcal{H}^{cl_\infty}_{\infty})$ and we denote

$$\Psi_0^\infty = e^{-iH_{\infty}t_0} \partial_t \Psi_\infty(t_0)$$  \hspace{1cm} (205)

its initial data at $t = 0$. From (196), (197), we get

$$\lim_{t \to +\infty} \| e^{iH_{\infty}t} \Psi_0^\infty - J^*_{\infty} \partial_t \left( e^{iH_0t} \Psi_0 \right) \|_{\mathcal{H}_{\infty}} = 0.$$  \hspace{1cm} (206)

The value of $\partial_t \left( e^{iH_0t} \Psi_0 \right)$ at $t = 0$ is $iH_0 \Psi_0$. $H_0$ is a self-adjoint operator with dense domain on $\mathcal{H}$, its point spectrum is empty and the spaces $\mathcal{H}_0^\pm, \mathcal{H}^{cl_\infty}_{\infty}$ are invariant under $H_0$. Therefore the direct sum of the sets

$$\left\{ H_0 \Psi_0; \ \Psi_0 \in \mathcal{H}_0^- \cap \mathcal{E}^{cl_\infty}_{\infty}; \ (l, n) \in \mathcal{I}_{\infty}; \ \varepsilon = +, - \right\}$$  \hspace{1cm} (207)

is dense in $\mathcal{H}_0^-$. (206) shows that for an initial data $H_0 \Psi_0$ in a set of type (207), the limit

$$\Psi_0^\infty = \lim_{t \to +\infty} e^{-iH_{\infty}t} J^*_{\infty} e^{iH_0t} H_0 \Psi_0$$  \hspace{1cm} (208)

exists in $\mathcal{H}_{\infty}$. The operator $W_0^\infty$ is consequently well-defined from $\mathcal{H}_0$ into $\mathcal{H}_{\infty}$. Since the local energy of the solution $e^{iH_0t} H_0 \Psi_0$ goes to zero when $|t|$ goes to $+\infty$, the limit $\Psi_0^\infty$ is independent of the choice of $\chi_{\infty}$ satisfying (65).

Q.E.D.

### 7 Conclusion

The scattering theory developed in this paper is only valid for the linear massless Dirac system. In the case of a massive field and when space-time is asymptotically flat, the mass of the field induces long-range perturbations at infinity and classical wave operators will probably not exist. However, using the methods developed by J. Dollard and G. Velo [10] and by V. Enss and B. Thaller [11] about the relativistic Coulomb scattering of Dirac fields as well as the works of A. Bachelot [1] and J. Dimock and B. Kay [9] on the Klein-Gordon equation on the Schwarzschild metric, it must be possible to show the existence and asymptotic completeness of Dollard-modified wave operators at infinity.

### References


